HAUSDORFF CENTER FOR MATHEMATICS

A probabilistic approach to diagram algebras

Jonas Wahl

Hausdorff Center for Mathematics,
University of Bonn

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Outline

1. What is a diagram algebra?
2. Branching graphs
3. The Vershik-Kerov boundary of a branching graph

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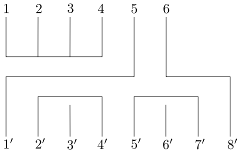
2. Branching graphs
3. The Vershik-Kerov boundary of a branching graph

- $\mathrm{Part}(k,l) = \{$Set partitions on k upper and l lower points $\},\ k,l \geq 0$.



A (noncrossing) set partition with 6 upper and 8 lower points.

- Operations on partitions:
  - involution (reflection along a horizontal line in the middle);
  - rotation;
  - tensor product (horizontal concatenation);
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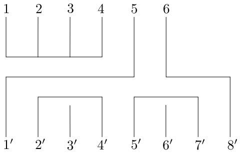

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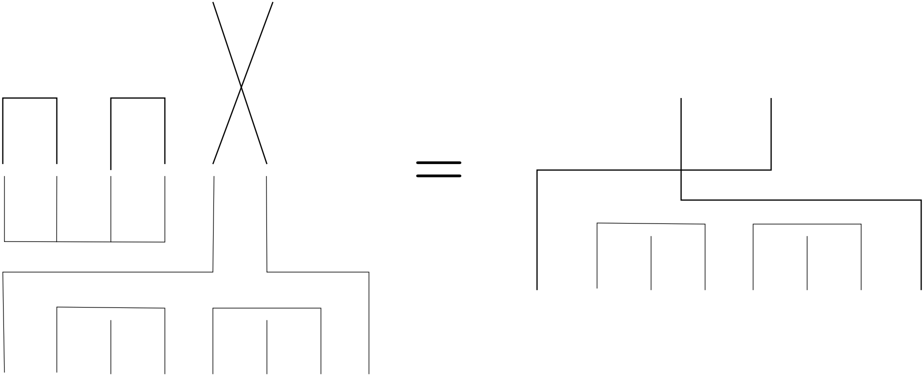
Multiplication of partitions



Multiplication of two partitions $p \in \mathcal{C}(2,6),\ q \in \mathcal{C}(6,8)$ yielding $p \cdot q \in \mathcal{C}(2,8)$.

- Category of partitions $\mathcal{C} = (\mathcal{C}(k,l))_{k,l\geq 0}$, such that
  - $\mathcal{C}(k,l) \subset \mathrm{Part}(k,l)$;
  - $| \in \mathcal{C}(1,1)$;
  - family is invariant under operations on partitions.
- Certain classes of categories of partitions have been classified (Banica-Speicher, Weber,...), for instance:
  - noncrossing/ planar categories: $\mathcal{S}^+ = \mathrm{NC}$, $\mathcal{O}^+ = \mathrm{NC}_2$, $\mathcal{B}^+ = \{$blocks of size one or two$\}$, $\mathcal{H}^+ = \{$blocks of even size$\}$, $\mathcal{S}^{+'}$, $\mathcal{B}^{+'}$, $\mathcal{B}^{+\#}$.
  - fully crossing categories (containing simple crossing  $\in \mathcal{C}(2,2)$): $\mathcal{S} = \mathrm{Part}$, $\mathcal{O} = \mathrm{Part}_2$, $\mathcal{B} = \{$blocks of size one or two$\}$, $\mathcal{H} = \{$blocks of even size$\}$, $\mathcal{S}'$, $\mathcal{B}'$.
  - halfliberated (containing  but not ): $\mathcal{O}^*, \mathcal{B}^{*\#}, \mathcal{H}^*$ and hyperoctahedral series.

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$\mathcal{C}$ = category of partitions, $\delta \in \mathbb{C}$.

- Diagram algebras of $(\mathcal{C}, \delta)$:
  - $A_{(\mathcal{C},\delta)}(k) = \{\sum_{p \in \mathcal{C}(k,k)} a_p e_p \ ; \ a_p \in \mathbb{C}\}$
    = complex free vector space with basis $\{e_p \ ; \ p \in \mathcal{C}(k,k)\}$.
  - Multiplication: $e_p \cdot e_q = \delta^{\#\text{erased blocks}} e_{p \cdot q}$.

- Many of the diagram algebras of the families from the previous slide have special names in the literature:
  - $A_{(\mathcal{O}^+,\delta)}(k) = \mathrm{TL}_\delta(k)$ **Temperley-Lieb algebras** (V. Jones '83).
  - $A_{(\mathcal{B}^+,\delta)}(k) = \mathrm{Mo}_\delta(k)$ **Motzkin algebras** (Benkart-Halverson '14).
  - $A_{(\mathcal{H}^+,\delta)}(k) = \mathrm{FC}_\delta(k)$ **Fuss-Catalan algebras** (Bisch-Jones '95).
  - $A_{(\mathcal{O},\delta)}(k) = \mathrm{Br}_\delta(k)$ **Brauer algebras** (Wenzl '88).
  - $A_{(\mathcal{B},\delta)}(k) = \mathrm{rBr}_\delta(k)$ **rook-Brauer algebras** (delMas-Halverson '13).
  - $A_{(\mathcal{S},\delta)}(k) = \mathrm{Part}_\delta(k)$ **Partition algebras** (Jones '94, Martin '96).
  - $A_{(\mathcal{O}^*,\delta)}(k) = \mathrm{wBr}_\delta(k)$ **walled Brauer algebras** (Nikitin '07, follows from Banica-Vergnioux '09).

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Diagram algebras play important roles in

- quantum groups;
- subfactors;
- knot theory;
- algebraic combinatorics (e.g. RSK algorithms);
- loop models in statistical physics:
  big open questions that can be formulated purely in terms of diagram algebras such as Razumov-Stroganov conjectures.

If the loop parameter δ is 'generic', we can define a direct limit C^*-algebra

$$A_{(\mathcal{C},\delta)}(\infty) = \lim_{\rightarrow} A_{(\mathcal{C},\delta)}(k)$$

since there are natural embeddings

$$A_{(\mathcal{C},\delta)}(0) \subset A_{(\mathcal{C},\delta)}(1) \subset A_{(\mathcal{C},\delta)}(2) \subset \dots$$

One can understand this tower of algebras through its *branching graph* or *Bratteli diagram*.

- Branching graph = (directed) graded graph Γ with vertex set $\bigcup_{k=0}^{\infty} \Gamma_k$ and edges from level k to level $k+1$.
- Get the following branching graph from our tower: where
  - $\Gamma_k = \{$ irreducible representations $\pi : A_{(\mathcal{C},\delta)}(k) \to L(V_\pi)\}$,
  - number of edges from $\rho \in \Gamma_{k-1}$ to $\pi \in \Gamma_k$ is the multiplicity of ρ in decomposition of $\pi|_{A_{(\mathcal{C},\delta)}(k-1)}$ into irreducibles.

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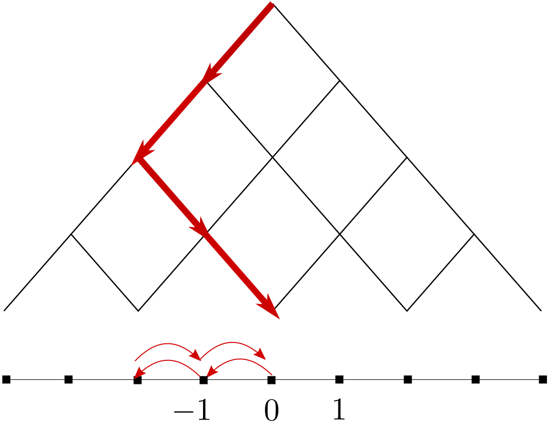
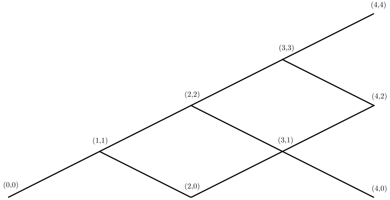
- Before approaching our diagram algebras, we have a look at the *Pascal graph*.

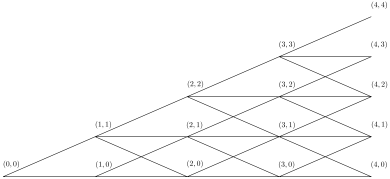


- Paths on Pascal graphs are trajectories of a walker on $\mathbb{Z}$ starting at 0.

- Branching graph of $\cdots \subset \mathrm{TL}_\delta(k) \subset \ldots$ = *semi-Pascal graph* (Jones):

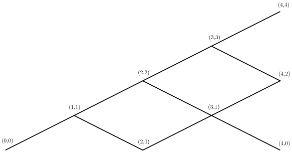


- Branching graph of $\cdots \subset \mathrm{Mo}_\delta(k) \subset \ldots$ (Halverson-Benkart, W.):



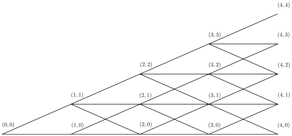
Paths on these graphs have several combinatorial interpretations:

- For $\dots \subset \mathrm{TL}_\delta(k) \subset \dots$:



  - *Ballot paths* on $\mathbb{N} \times \mathbb{N}$ (allowed steps $(+1,+1),\ (+1,-1)$).
  - Walks on the half-line $\mathbb{N}$.

- For $\dots \subset \mathrm{Mo}_\delta(k) \subset \dots$:



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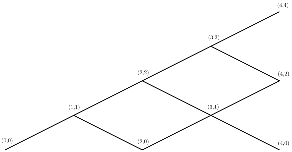

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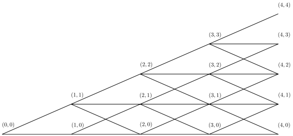

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- All branching graphs of noncrossing or fully crossing diagram algebras can be obtained as graphs of walks on smaller *principal graphs*.

Definition (Vershik-Nikitin '06)

Let Λ be a branching graph. Denote by $|\lambda|$ the level of the vertex λ in Λ. The **pascalization** $\mathcal{P}(\Lambda)$ is the branching graph with

- vertex level sets $\mathcal{P}(\Lambda)_n = \{(n,\lambda)\ ;\ |\lambda| \leq n,\ |\lambda| = n \bmod 2\}$;
- an edge $(n,\lambda) \nearrow_{\mathcal{P}(\Lambda)} (n+1,\tilde{\lambda})$ for every edge between λ and $\tilde{\lambda}$ ($\lambda \nearrow \tilde{\lambda}$ or $\tilde{\lambda} \nearrow \lambda$).

- Every branching graph Γ of a tower of diagram algebras is the pascalization $\Gamma = \mathcal{P}(\Lambda)$ of a principal graph Λ.
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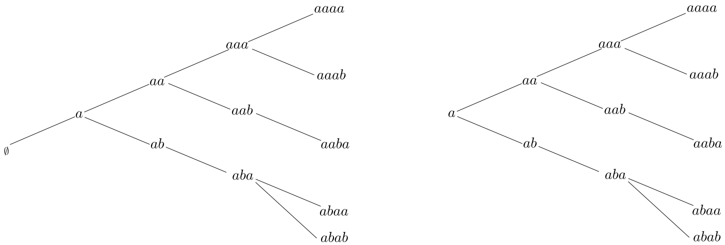
- Principal graph for

$$\cdots \subset A_{(\mathcal{H}^{+},\delta)}(k) \subset A_{(\mathcal{H}^{+},\delta)}(k) \subset \ldots$$

and

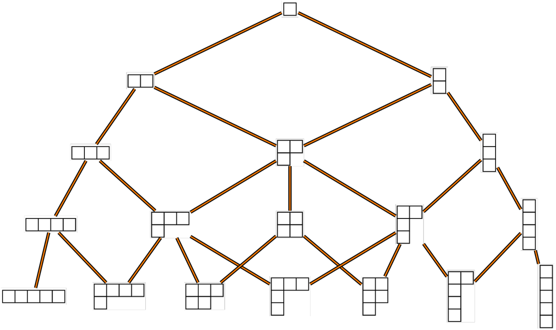
$$\cdots \subset A_{(\mathcal{B}^{+\#},\delta)}(k) \subset A_{(\mathcal{B}^{+\#},\delta)}(k+1) \subset \ldots$$



Fibonacci tree $(\mathcal{H}^{+})$ and derooted Fibonacci tree $(\mathcal{B}^{+\#})$.

- For the diagram algebras with the simple crossing, the principal graphs are variations of the *Young graph* $\mathbb{Y}$, the branching graph of

$$\{e\} = S_1 \subset S_2 \subset S_3 \subset \dots$$



The Young graph

- For the Brauer algebras $(\mathrm{Br}_\delta(k))_k$ (pair partitions), the principal graph is $\mathbb{Y}$ (Vershik-Nikitin);
- For $(\mathrm{Part}_\delta(k))_k$, the principal graph is $\bar{\mathbb{Y}}$ ($\mathbb{Y}$, but every level is repeated twice) (Vershik-Nikitin);
- For the rook-Brauer algebras $(\mathrm{rBr}_\delta(k))_k$ (pair partitions and singletons), paths on the branching graph are again *lazy* walks on $\mathbb{Y}$ (W. '20);
- For the algebras $(A_{(\mathcal{H},\delta)}(k))_k$, the principal graph is the *coupled Young graph* (W. 20):
  - vertices = pairs of Young diagrams (μ, λ);
  - growth rule:

$$(\mu, \lambda) \nearrow (\mu + \square, \lambda) \quad \text{or} \quad (\mu, \lambda) \nearrow (\mu - \square, \lambda + \square).$$

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- How can you move randomly down a branching graph in such a way that your past does not matter? $\rightsquigarrow$ central measures.
- Let Γ be a branching graph, $\tilde{\gamma} \in \Gamma_m,\ \gamma \in \Gamma_n,\ n \geq m$. Then

$$\dim_\Gamma(\tilde{\gamma};\gamma) := \#\text{ paths from } \tilde{\gamma} \text{ to } \gamma \qquad \dim_\Gamma(\gamma) = \dim_\Gamma(\emptyset;\gamma).$$

Definition

A measure $\mathbb{P}$ on the space of infinite rooted paths $\Omega = \{\emptyset = \omega_0 \nearrow \omega_1 \nearrow \omega_2 \dots\} \subset \prod_{n=0}^{\infty} \Gamma_n$ is **central** if for every path $\gamma_0 = \emptyset \nearrow \gamma_1 \nearrow \dots, \gamma_n = \gamma$ from the root to γ, we have

$$\mathbb{P}\left(\{\omega = (\omega_0 \nearrow \omega_1 \nearrow \dots) \in \Omega\,;\, \omega_1 = \gamma_1, \dots, \omega_n = \gamma\}\right) = \frac{\mathbb{P}(\{\omega_n = \gamma\})}{\dim_\Gamma(\gamma)}.$$

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- The set of central measures is convex and, more precisely forms a **Choquet simplex**, i.e. every central measure can be uniquely represented by a probability measure over its *extreme* points.
- The **trace simplex** on $A_{(\mathcal{C},\delta)}(\infty)$ is homeomorphic to the simplex of central measures on the associated branching graph Γ:
  - Want to define τ = trace on $A_{(\mathcal{C},\delta)}(\infty)$ given central measure $\mathbb{P}$;
  - τ is determined by restrictions $\tau|_{A_{(\mathcal{C},\delta)}(k)}$ for all k;
  - Decompose $A_{(\mathcal{C},\delta)}(k) = \bigoplus_{\gamma\in\Gamma_k} M_{\dim_\Gamma(\gamma)}(\mathbb{C})$ as a sum of matrix algebras;
  - Set $\tau_\gamma =$ regular trace on $M_{\dim_\Gamma(\gamma)}(\mathbb{C})$;
  - Define $\tau|_{A_{(\mathcal{C},\delta)}(k)} = \sum_{\gamma\in\Gamma_k} \mathbb{P}(\{\omega_k = \gamma\}) \frac{\tau_\gamma}{\dim_\Gamma(\gamma)}$.
- Minimal boundary of branching graph Γ:

$$\partial\Gamma := \{\text{extremal central measures}\} \cong \{\text{ extremal traces on limit algebra }\}$$

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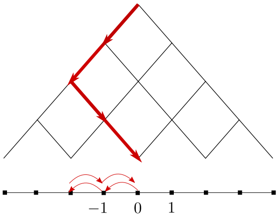

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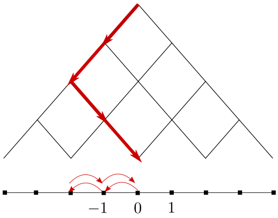

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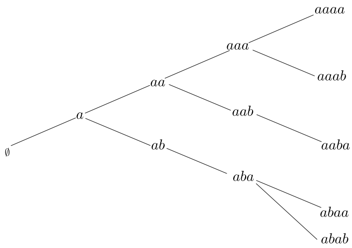

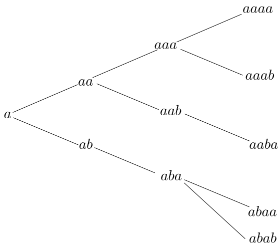

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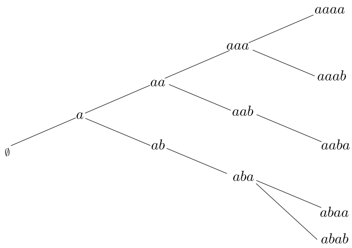

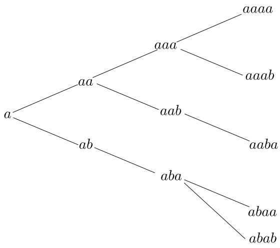

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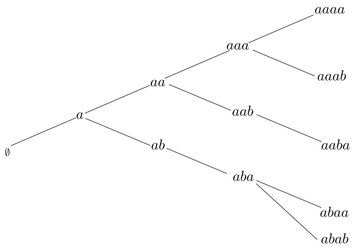

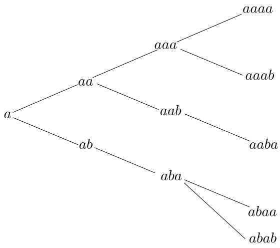

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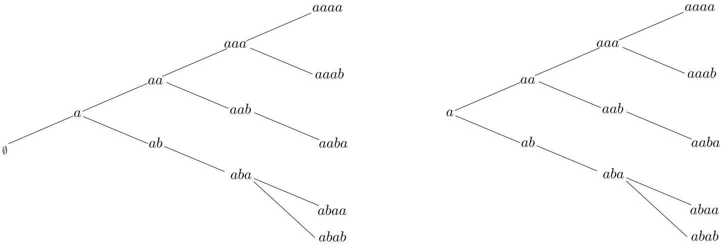

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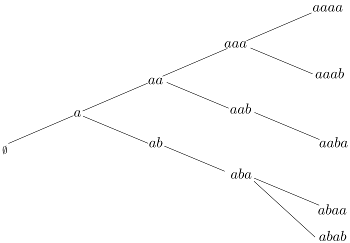

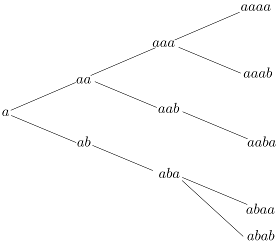

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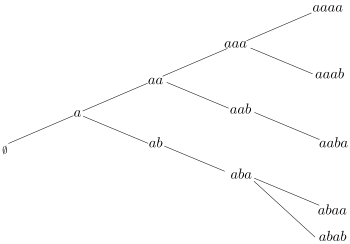

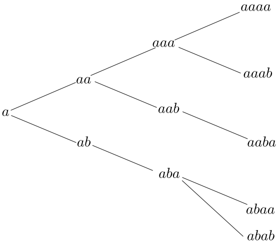

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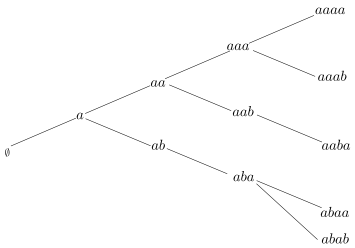

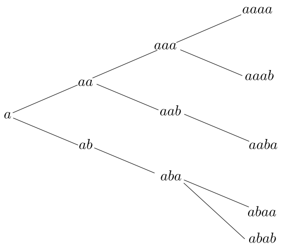

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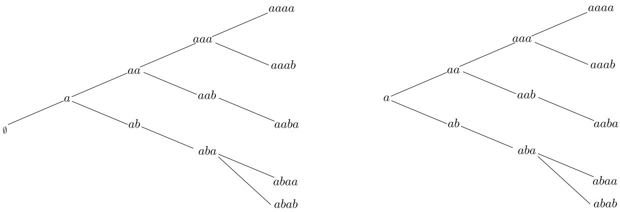

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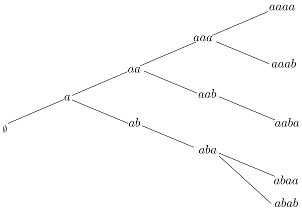

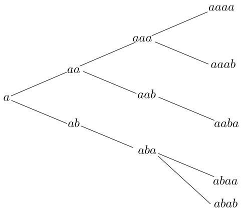

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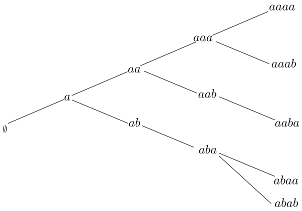

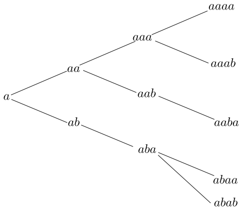

- The random walk $S_{(t,\eta)}$ induces a central measure $\mathbb{P}_{(t,\eta)}$ on $\mathcal{P}(\mathbb{FT})$ with transition probabilities

$$\tilde{p}_{(t,\eta)}((n,v),(n+1,w)) = p_{(t,\eta)}(v,w).$$

Theorem (W.20)

The extremal central measures on $\mathcal{P}(\mathbb{FT})$ are exactly the measures $\mathbb{P}_{(t,\eta)}$, i.e. the minimal boundary is

$$\partial\mathcal{P}(\mathbb{FT}) = \{\mathbb{P}_{(t,\eta)}\ ;\ t \in \mathrm{Ends}(\mathbb{FT}),\ \eta \in [0,4/27].\}$$

Ingredients of the proof:

- Use ergodic method and walk counting on $\mathbb{FT}$ to see that extremal measures must be of the form $\mathbb{P}_{(t,\eta)}$.
- To prove extremality, show that $S_{(t,\eta)}$ converges almost surely to t.
- Prove a law of large numbers for the times of last exit at vertices of t.

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$$\tilde{p}_{(t,\eta)}((n,v),(n+1,w)) = p_{(t,\eta)}(v,w).$$

Theorem (W.20)

The extremal central measures on $\mathcal{P}(\mathbb{FT})$ are exactly the measures $\mathbb{P}_{(t,\eta)}$, i.e. the minimal boundary is

$$\partial\mathcal{P}(\mathbb{FT}) = \{\mathbb{P}_{(t,\eta)}\ ;\ t \in \mathrm{Ends}(\mathbb{FT}),\ \eta \in [0,4/27].\}$$

Ingredients of the proof:

- Use ergodic method and walk counting on $\mathbb{FT}$ to see that extremal measures must be of the form $\mathbb{P}_{(t,\eta)}$.
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- Recall: branching graphs are pascalizations $\Gamma = \mathcal{P}(\Lambda)$ where Λ is closely related to the Young graph $\mathbb{Y}$;
- In fact: can show that the boundary of the principal graph Λ is the same as the boundary of $\mathbb{Y}$, i.e. the Thoma simplex;
- For categories of partitions $\mathcal{S}$ (all partitions), $\mathcal{O}$ (pair partitions), $\mathcal{B}$ (pairs and singletons): boundary of Γ is fully supported on principal graph Λ and is therefore also the Thoma simplex T;
- For $\mathcal{H}$ still open, but strong empirical evidence that this is true (combinatorics much harder);
- For category of partitions $\mathcal{O}^*$ dual to *halfliberated* orthogonal group: boundary of branching graph Γ is $T \times T$ (follows from results of Banica-Vergnioux and Vershik-Nikitin).
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Thanks for

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