UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Set Theory and Forcing*
Summer term 2019

Sheet 1
to be discussed on Tuesday, 23 April, 2019 in SR6 (room 217, E2 4)

Exercise 1. Check that the following functions are primitive recursive.

(a) $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (a, b) \mapsto a + b$ and $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (a, b) \mapsto ab$

(b) $\mathrm{sg} : \mathbb{N} \to \mathbb{N}$ and $\overline{\mathrm{sg}} : \mathbb{N} \to \mathbb{N}$ and $\mathrm{pd} : \mathbb{N} \to \mathbb{N}$ with

$$\mathrm{sg}(0) := 0, \mathrm{sg}(x + 1) := 1, \quad \overline{\mathrm{sg}}(0) := 1, \overline{\mathrm{sg}}(x + 1) := 0, \quad \mathrm{pd}(0) := 0, \mathrm{pd}(x + 1) := x$$

(c) $- : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (a, b) \mapsto a - b$ if $a \geq b$ and $(a, b) \mapsto 0$ otherwise. (Use pd.)

(d) For $P \in$PRP^{n+1} the function $\mu^b P : \mathbb{N}^{n+1} \to \mathbb{N}$ given by

$$\mu^b P(k, x_1, \ldots, x_n) := \begin{cases} \min\{y \leq k; \mid P(y, x_1, \ldots, x_n)\} & \text{if such a } y \text{ exists} \\ k + 1 & \text{otherwise} \end{cases}$$

Exercise 2. Prove Thm. 1.10 of the lecture: The class PRP is closed under the following.

(a) If $P, Q \in$PRP, then $\neg P, P \cap Q, P \cup Q \in$PRP. (Use Ex. 1(b).)

(b) If $P \in$PRP^{n+1}, then $\exists^b P, \forall^b P \in$PRP^{n+1}. (Use Ex. 1(d).)

(c) If $P \in$PRP^m and $f_1, \ldots, f_m \in$PRF^n, then $\{x \in \mathbb{N}^n \mid P(f_1(x), \ldots, f_m(x))\} \in$PRP^n.

Exercise 3. The *Ackermann function* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by:

$$f(0, y) := y + 1, \qquad f(x + 1, 0) := f(x, 1), \qquad f(x + 1, y + 1) := f(x, f(x + 1, y))$$

Just for fun, try to find the value $f(4, 2)$ in the internet (but not in the universe).

(a) If you want, check:

1. $f(1, y) = y + 2$ and $f(2, y) = 2y + 3$
2. $f(x, y) > x + y$ and $f(x, y + 1) > f(x, y)$ and $f(x + 1, y) \geq f(x, y + 1)$
3. For all $x_1, \ldots, x_n$ there is an x such that for all y: $\sum_{i=1}^n f(x_i, y) \leq f(x, y)$.

(b) Prove that for any $g \in$PRF, there is an x with $g(x_1, \ldots, x_n) < f(x, \sum_{i=1}^n x_i)$.

(c) Check that $f_n := f(n, \cdot) \in$PRF for any $n \in \mathbb{N}$, but $f \notin$PRF.

Hints: Use (a) and the inductive definition of PRT for (b); use Thm. 1.14 for $f_n \in$PRF; use (b) for $f \notin$PRF.