UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Prof. Dr. Moritz Weber

Exercises for the lecture *Set Theory and Forcing*
Summer term 2019

Sheet 2
to be discussed on Tuesday, 7 May, 2019 in SR6 (room 217, E2 4)

Recall that $\mathbb{F}$ is the class of all total functions in $\mathbb{P}$ (Def. 2.3). Given $P \subset \mathbb{N}^n$, we say that

- P is *recursive*, if its characteristic function χ_P is in $\mathbb{F}$,
- P is *recursively enumerable*, if there is a function $f : \mathbb{N} \to \mathbb{N}^n$ whose components $f_1, \dots, f_n$ are in $\mathbb{F}$ and P is the range of f.

In Thm. 2.18 we will show that P is recursively enumerable if and only if there is a recursive $R \subset \mathbb{N}^{n+1}$ with:

$$P(x_1, \dots, x_n) \qquad \Longleftrightarrow \qquad \exists y : R(x_1, \dots, x_n, y)$$

Given T^n, the universal predicate of Kleene's Normal Form Theorem (Thm. 2.13), and $e \in \mathbb{N}$, we put:

$$W_e^n := \{x = (x_1, \dots, x_n) \in \mathbb{N}^n \mid \exists y : T^n(e, x, y)\}$$

Exercise 1. Let $e, n \in \mathbb{N}$, $n \neq 0$. Prove the following statements.

(a) W_e^n is recursively enumerable.

(b) W_e^n is universal for all recursively enumerable predicates, i.e. if $P \subset \mathbb{N}^n$ is recursively enumerable, then there is $e \in \mathbb{N}$ such that $P = W_e^n$.

Exercise 2. We put $W_e := W_e^1$.

(a) Show that there is $e \in \mathbb{N}$ such that $W_e = \{3n \mid n \in \mathbb{N}\}$.

(b) Show that there is $e \in \mathbb{N}$ such that $W_e = \{e\}$. (Use the second Recursion Theorem Cor. 2.14: $\forall f \in \mathbb{P}^{n+1}\ \exists e \in \mathrm{PI} : \{e\}(x_1, \dots, x_n) \simeq f(e, x_1, \dots, x_n)$.)

Exercise 3. Use the second Recursion Theorem (Cor. 2.14) in order to show that the Ackermann function f is in $\mathbb{P}$. We conclude $f \in \mathbb{P} \setminus \mathrm{PRF}$.