UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Prof. Dr. Moritz Weber

Exercises for the lecture *Set Theory and Forcing*
Summer term 2019

Sheet 3
to be discussed on Tuesday, 21 May, 2019 in SR6 (room 217, E2 4)

The *propositional parts* $PP(F)$ of an $\mathcal{L}$-formula F are:

- If F is an atom formula $F = Pt_1 \dots t_n$, then $PP(F) = \{F\}$.
- If $F = \neg G$, then $PP(F) = \{F\} \cup PP(G)$.
- If $F = G_1 \circ G_2$, with $\circ \in \{\wedge, \vee\}$, then $PP(F) = \{F\} \cup PP(G_1) \cup PP(G_2)$.
- If $F = QxG_u(x)$, with $Q \in \{\forall, \exists\}$, then $PP(F) = \{F\}$.

Let PA be the set of all formulas F with $PP(F) = \{F\}$. A *Boolean interpretation* is a map $\mathbb{B} : PA \to \{\text{false}, \text{true}\}$. For $F \notin PA$, we put:

- If $F = \neg G$, then $\mathbb{B}(F) = \neg\mathbb{B}(G)$.
- If $F = G_1 \circ G_2$, with $\circ \in \{\wedge, \vee\}$, then $\mathbb{B}(F) = (\mathbb{B}(G_1) \circ \mathbb{B}(G_2))$.
- If $F = G_1 \to G_2$, then $\mathbb{B}(F) = (\mathbb{B}(G_1) \Rightarrow \mathbb{B}(G_2))$.

If $\mathbb{B}(F) = \text{true}$ for all Boolean interpretations, we write $\vDash_b F$.

Exercise 1. (a) If $\mathcal{S}$ and Φ is an $\mathcal{L}$-interpretation, show that $\mathbb{B}_\Phi^{\mathcal{S}}(F) := F^{\mathcal{S}}[\Phi]$ defines a Boolean interpretation.

(b) Using (a), prove that $\vDash_b F$ implies $\vDash F$.

(c) Give examples of formulas F with $\vDash_b F$, and counter examples for the converse direction in (b).

Exercise 2. Let $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ be an embedding (Def. 6.6). Let $\mathcal{L}_{S_1}$ be the extension of the language $\mathcal{L}$ by adding symbols $\underline{s}$ for all elements $s \in S_1$. Let F be an $\mathcal{L}_{S_1}$ theorem.

(a) Let F be quantifier free. Show: $\mathcal{S}_1 \vDash F(\underline{s_1}, \dots, \underline{s_n}) \iff \mathcal{S}_2 \vDash F(\underline{\varphi s_1}, \dots, \underline{\varphi s_n})$

(b) Let φ be an elementary embedding. Show: $\mathcal{S}_1 \vDash F(\underline{s_1}, \dots, \underline{s_n}) \iff \mathcal{S}_2 \vDash F(\underline{\varphi s_1}, \dots, \underline{\varphi s_n})$

Exercise 3. Ask questions about some details of Gödel's Incompleteness Theorems.