UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
Prof. Dr. Roland Speicher
M.Sc. Felix Leid

Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 10
Hand in on Wednesday, 23.01.2019, before the lecture.

Exercise 1 (2 weeks , 20 points).

(i) Let A_N be a GUE(N) and D_N a deterministic $N \times N$-matrix, such that D_N converges to d where
$$\mu_d = \frac{1}{2}(\delta_{-1} + \delta_{+1}).$$
We know that then the eigenvalue distribution of $A_N + D_N$ converges to $\mu_s \boxplus \mu_d$. Check this by comparing the density of $\mu_s \boxplus \mu_d$ with histograms of matrices $A_N + D_N$ for large N.

(ii) Let U_N be a Haar unitary $N \times N$ random matrix. Let A_N and B_N be deterministic $N \times N$ matrices such that A_N converges to a and B_N converges to b, where
$$\mu_a = \frac{1}{2}(\delta_{-1} + \delta_{+1}), \qquad \mu_b = \frac{1}{4}(\delta_{-1} + 2\delta_0 + \delta_{+1}).$$
We know that then the eigenvalue distribution of $U_N A_N U_N^* + B_N$ converges to $\mu_a \boxplus \mu_b$. Check this by comparing the density of $\mu_a \boxplus \mu_b$ with eigenvalue histograms of matrices $U_N A_N U_N^* + B_N$ for large N.

Exercise 2 (10 points).
Calculate the $*$-cumulants of a Haar unitary u. (For this compare also Exercise 3 on Assignment 6.)

Exercise 3 (10 points).
For permutations $\alpha, \beta \in S_m$ we define:
$$|\alpha| := m - \#\alpha, \qquad d(\alpha, \beta) := |\beta\alpha^{-1}|$$

(i) Show that $|\alpha|$ is equal to the minimal non-negative integer k such that α can be written as a product of k transpositions.

(ii) Show that $|\cdot|$ satisfies: $|\alpha\beta| \leq |\alpha| + |\beta|$ for all $\alpha, \beta \in S_m$.

(iii) Show that d is a distance (or metric).

Exercise 4 (10 points).
We put for $\gamma = (1, 2, \dots, m-1, m) \in S_m$

$$S_{NC}(\gamma) := \{\alpha \in S_m \mid d(e, \alpha) + d(\alpha, \gamma) = m - 1 = d(e, \gamma)\};$$

i.e., elements in $S_{NC}(\gamma)$ are those permutations in S_m which lie on a geodesic from the identity element e to the long cycle γ. One can show that $S_{NC}(\gamma)$ is canonically isomorphic to $NC(m)$; hence one has an embedding from $NC(m)$ into the permutation group S_m.

(i) Identify this embedding for $m = 1, 2, 3, 4$.

(ii) Check by a non-trivial example for $m = 4$ that under this embedding a pair σ, π in $NC(m)$ with $\sigma \leq \pi$ is mapped to a pair α, β in S_m with $|\alpha^{-1}\beta| + |\alpha| + |\beta^{-1}\gamma| = m-1$.