UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 1
Hand in on Monday, 29.10.18, before the lecture.

Exercise 1 (10 points).
We consider, for a group G, the non-commutative probability space $(\mathbb{C}G, \tau_G)$ from Example 1.2.

(i) Show that τ_G is tracial.

(ii) We define on $\mathbb{C}G$ the anti-linear mapping * by

$$(\sum_g \alpha_g g)^* := \sum_g \bar{\alpha}_g g^{-1}.$$

Show that $(\mathbb{C}G, \tau_G)$ becomes a $*$-probability space with respect to this structure, i.e. that τ_G is positive.

(iii) Show that τ_G is faithful.

Exercise 2 (10 points + 5 extra points*).

(i) Consider random variables a_1, a_2, b_1, b_2 such that $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free (which means that $a_1, a_2, b_1, b_2 \in \mathcal{A}$ for some non-commutative probability space $(\mathcal{A}, \varphi)$ such that the unital algebra generated by a_1 and a_2 is free from the unital algebra generated by b_1 and b_2). Show, from the definition of freeness that we have then

$$\varphi(a_1b_1a_2b_2) = \varphi(a_1a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1b_2) - \varphi(a_1)\varphi(b_1)\varphi(a_2)\varphi(b_2).$$

(ii)* Try to find a formula for $\varphi(a_1b_1a_2b_2a_3)$, if $\{a_1, a_2, a_3\}$ and $\{b_1, b_2\}$ are free. Think about how much time it would take you to calculate a formula for $\varphi(a_1b1a_2b_2a_3b_3)$, if $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b3\}$ are free.

Exercise 3 (10 points).

(i) Prove that functions of freely independent random variables are freely independent; more precisely: if a and b are freely independent and f and g are polynomials, then $f(a)$ and $g(b)$ are freely independent, too.

(ii) Assume that $(\mathcal{A}, \varphi)$ is a $*$-probability space and φ is faithful. Show the following: If two unital $*$-subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are freely independent, then

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \mathbb{C}1.$$