UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 2
Hand in on Wednesday, 31.10.18, before the lecture.

Exercise 1 (10 points).
In this exercise we prove that free independence behaves well under successive decompositions and thus is associative. Consider a non-commutative probability space $(\mathcal{A}, \varphi)$. Let $(\mathcal{A}_i)_{i\in I}$ be unital subalgebras of $\mathcal{A}$ and, for each $i \in I$, $(\mathcal{B}^i_j)_{j\in J(i)}$ unital subalgebras of $\mathcal{A}_i$. Denote the restriction of φ to $\mathcal{A}_i$ by φ_i. Note that then $(\mathcal{A}_i, \varphi_i)$ is, for each $i \in I$, a non-commutative probability space on its own. Then we have:

(i) If $(\mathcal{A}_i)_{i\in I}$ are freely independent in $(\mathcal{A}, \varphi)$ and, for each $i \in I$, $(\mathcal{B}^i_j)_{j\in J(i)}$ are freely independent in $(\mathcal{A}_i, \varphi_i)$, then all $(\mathcal{B}^i_j)_{i\in I; j\in J(i)}$ are freely independent in $(\mathcal{A}, \varphi)$.

(ii) If all $(\mathcal{B}^i_j)_{i\in I; j\in J(i)}$ are freely independent in $(\mathcal{A}, \varphi)$ and if, for each $i \in I$, $\mathcal{A}_i$ is as algebra generated by all $\mathcal{B}^i_j$ for $j \in J(i)$, then $(\mathcal{A}_i)_{i\in I}$ are freely independent in $(\mathcal{A}, \varphi)$.

Prove one of those two statements!

Exercise 2 (10 points).
Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Consider a unital subalgebra $\mathcal{B} \subset \mathcal{A}$ and a Haar unitary $u \in \mathcal{A}$, such that $\{u, u^*\}$ and $\mathcal{B}$ are free. Show that then also $\mathcal{B}$ and $u^*\mathcal{B}u$ are free, where

$$u^*\mathcal{B}u := \{u^*bu \mid b \in \mathcal{B}\}.$$

Remark: A Haar unitary is a unitary $u \in \mathcal{A}$, i.e. $u^*u = 1 = uu^*$, which satisfies $\varphi(u^k) = \delta_{0,k}$ for any $k \in \mathbb{Z}$.

Exercise 3 (10 points).
Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a_i \in \mathcal{A}$ $(i \in I)$ be free. Consider a product $a_{i(1)} \cdots a_{i(k)}$ with $i(j) \in I$ for $j = 1, \dots, k$. Put $\pi := \ker(i(1), \dots, i(k)) \in \mathcal{P}(k)$. Show the following.

(i) We can write $\varphi(a_{i(1)} \cdots a_{i(k)})$ as a polynomial in the moments of the a_i, where each summand contains at least $\#\pi$ many factors.

(ii) If π is crossing then $\varphi(a_{i(1)} \cdots a_{i(k)})$ can be written as a polynomial in moments of the a_i, where each summand contains at least $\#\pi + 1$ many factors.