UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
Prof. Dr. Roland Speicher
M.Sc. Felix Leid

Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 3
Hand in on Monday, 7.11.18, before the lecture.

Exercise 1 (10 points).
Let $f(z) = \sum_{m=0}^{\infty} C_m z^m$ be the generating function (considered as formal power series) for the numbers $\{C_m\}_{m\in\mathbb{N}_0}$, where the C_m are defined by $C_0 = 1$ and by the recursion

$$C_m = \sum_{k=1}^{m} C_{k-1}C_{m-k}, \qquad (m \geq 1).$$

(i) Show that
$$1 + zf(z)^2 = f(z).$$

(ii) Show that f is also the power series for
$$\frac{1-\sqrt{1-4z}}{2z}.$$

(iii) Show that
$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

Exercise 2 (10 points).
Show that the moments of the semicircular distribution are given by the Catalan numbers; i.e., for $m \in \mathbb{N}_0$
$$\frac{1}{2\pi}\int_{-2}^{+2} t^{2m}\sqrt{4-t^2} = \frac{1}{m+1}\binom{2m}{m}.$$

Exercise 3 (10 points + 5 points*).
Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space with orthonormal basis $(e_n)_{n=0}^{\infty}$. On $B(\mathcal{H})$ we define the state
$$\varphi(a) = \langle e_0, ae_0\rangle.$$
We consider now the creation operator $l \in B(\mathcal{H})$ which is defined by linear and continuous extension of
$$le_n = e_{n+1}.$$

please turn over

(i) Show that its adjoint ("annihilation operator") is given by extension of

$$l^* e_n = \begin{cases} e_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

(ii) Show that the operator $x = l + l^*$ is in the $*$-probability space $(B(\mathcal{H}), \varphi)$ a standard semicircular element.

(iii)* Is φ faithful on $B(\mathcal{H})$? How about φ restricted to the unital algebra generated by x?

Exercise 4 (10 points).
Let $(\mathcal{A}_n, \varphi_n)$ $(n \in \mathbb{N})$ and $(\mathcal{A}, \varphi)$ be non-commutative probability spaces. Let $(b_n^{(1)})_{n\in\mathbb{N}}$ and $(b_n^{(2)})_{n\in\mathbb{N}}$ be two sequences of random variables $b_n^{(1)}, b_n^{(2)} \in \mathcal{A}_n$ and let $b^{(1)}, b^{(2)} \in \mathcal{A}$. We say that $(b_n^{(1)}, b_n^{(2)})$ converges in distribution to $(b^{(1)}, b^{(2)})$, if we have the convergence of all joint moments, i.e., if

$$\lim_{n\to\infty} \varphi_n[b_n^{(i_1)} b_n^{(i_2)} \cdots b_n^{(i_k)}] = \varphi(b^{(i_1)} b^{(i_2)} \cdots b^{(i_k)})$$

for all $k \in \mathbb{N}$ and all $i_1, \ldots, i_k \in \{1, 2\}$.
Consider now such a situation where $(b_n^{(1)}, b_n^{(2)})$ converges in distribution to $(b^{(1)}, b^{(2)})$. Assume in addition that, for each $n \in \mathbb{N}$, $b_n^{(1)}$ and $b_n^{(2)}$ are free in $(\mathcal{A}_n, \varphi_n)$. Show that then freeness goes over to the limit, i.e., that also $b^{(1)}$ and $b^{(2)}$ are free in $(\mathcal{A}, \varphi)$.