UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 4
Hand in on Wednesday, 14.10.18, before the lecture.

Exercise 1 (10 points).
Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let the unital subalgebras $\mathcal{A}_i$ $(i \in I)$ be freely independent. Assume also that $\mathcal{A}$ is generated as an algebra by the $\mathcal{A}_i$. Show the following: If, for each $i \in I$, $\varphi|_{\mathcal{A}_i}$ is a homomorphism on $\mathcal{A}_i$ then φ is a homomorphism on $\mathcal{A}$.
Remark: The map $\varphi : \mathcal{A} \to \mathbb{C}$ is a homomorphism if it is linear and multiplicative in the sense $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$.

Exercise 2 (10 points).
Prove that $\#NC(k) = C_k$ by showing that $\#NC(k)$ satisfies the recursion relation for the Catalan numbers.

Exercise 3 (10 points).
Define, for $n \in \mathbb{N}_0$, the Bell numbers by $B_n := \#\mathcal{P}(n)$, where $B_0 = 1$.

(i) Show that the Bell numbers satisfy the recursion:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

(ii) Show that the Bell numbers are also the moments of a Poisson distribution of parameter $\lambda = 1$, i.e., that

$$B_n = \frac{1}{e} \sum_{p=0}^{\infty} p^n \frac{1}{p!}.$$