UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 5
Hand in on Wednesday, 21.10.18, before the lecture.

Exercise 1 (10 points)**.**
Let $f, g\colon \mathbb{N} \to \mathbb{C}$ be functions and define the Dirichlet convolution by

$$f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

We call a $f\colon \mathbb{N} \to \mathbb{C}$ multiplicative if $f(mn) = f(m)f(n)$ for all n, m with $\gcd(n, m) = 1$ and define functions $\chi\colon \mathbb{N} \to \mathbb{C}$ by $\chi(n) = 1$ for all $n \in \mathbb{N}$ and $\delta : \mathbb{N} \to \mathbb{C}$ by

$$\delta(n) = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases};$$

furthermore, we define the (number theoretic) Möbius function $\mu\colon \mathbb{N} \to \mathbb{C}$ by

- $\mu(1) = 1$
- $\mu(n) = 0$ if n is has a squared prime factor.
- $\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$ and all primes p_i are different.

Note that χ, δ, and μ are multiplicative. Show the following.

(i) Let f and g be multiplicative, show that $f * g$ is multiplicative.

(ii) Show that μ solves the inversion problem

$$g = f * \chi \iff f = g * \mu.$$

[Hint: Show that $\mu * \chi = \delta$.]

(iii) Show that Euler's phi function satisfies $\varphi = f * \mu$, where f is the identity function, i.e., $f(n) = n$ for all n.
Remark: the only information you need about Euler's phi function is that

$$\sum_{d|n} \varphi(d) = n$$

Exercise 2 (10 points)**.**
Let $P = B_n$ be the poset of subsets of $[n] = \{1, \ldots, n\}$, where $T \leq S$ means that $T \subset S$. (Note that $T \subset S$ includes also the case $T = S$.)

bitte wenden

(i) Show that the Möbius function of this poset is given by

$$\mu(T,S) = (-1)^{\#S-\#T} \qquad\qquad (T \subset S \subset [n]).$$

(ii) Conclude from Möbius inversion on this poset B_n the following inclusion-exclusion principle: Let X be a finite set and $X_1, \ldots, X_n \subset X$. Then we have

$$\#(X_1 \cup \cdots \cup X_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \#(X_{i_1} \cap \cdots \cap X_{i_k}).$$

[Hint: Consider the functions

$$f(I) = \#\left(\bigcap_{i \in I} X_i\right) \quad \text{and} \quad g(I) = \#\{x \in X \mid x \in X_i \ \forall i \in I; x \notin X_j \ \forall j \notin I\}$$

You might also assume that $X = X_1 \cup \cdots \cup X_n$.]

Exercise 3 (10 points).
In this exercise we want to introduce another example of an $*$-probability space containing a semicircular random variable, namely the *full Fock space*. Let $(V, \langle \cdot, \cdot \rangle)$ a inner product space over $\mathbb{C}$ and $\Omega \in V$ be a unit vector, called vacuum vector. Then we define the *full Fock space* by

$$\mathcal{F}(V) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

and an inner product by extension of

$$\langle v_1 \otimes \cdots \otimes v_i, w_1 \otimes \cdots \otimes w_j \rangle_{\mathcal{F}(V)} = \delta_{ij} \langle v_1, w_1 \rangle \ldots \langle v_i, w_i \rangle.$$

Defining the vacuum expectation

$$\varphi \colon \mathcal{A} := \mathrm{End}(\mathcal{F}((V)) \to \mathbb{C}, \quad a \mapsto \langle \Omega, a\omega \rangle_{\mathcal{F}(V)}$$

makes $(\mathcal{A}, \varphi)$ into a $*$-probability space Let $v \in V$ be a non-zero vector, then we define a linear operator on $\mathcal{F}(V)$ by

$$l(v)\Omega = v, \quad l(v) v_1 \otimes \cdots \otimes v_n = v \otimes v_1 \otimes \cdots \otimes v_n$$

and its adjoint by

$$l^*(v)\Omega = 0, \quad l^*(v) v_1 \otimes \cdots \otimes v_n = \langle v, v_1 \rangle v_2 \otimes \cdots \otimes v_n.$$

(i) Show that $x(x) = l(v) + l^*(v)$ is a semicircular element in $(\mathcal{A}, \varphi)$, if v is a unit vector.

(ii) Let $U_1 \perp U_2$ orthogonal subspaces in V and $u_i \in U_i$ unit vectors. Show that $l(u_1)$ and $l(u_2)$ are $*$-free, i.e. $\mathrm{alg}(1, l(u_1), l^*(u_1))$ and $\mathrm{alg}(1, l(u_2), l^*(u_2))$ are free.

Exercise 4.
Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

(i) Assume that φ is a trace, show that κ_n is invariant under cyclic permutations, i.e.

$$\kappa_n(a_1, a_2, \ldots, a_n) = \kappa(a_2, \ldots a_n, a_1)$$

for all $a_1, \ldots, a_n \in \mathcal{A}$.

(ii) Assume that φ is invariant under all permutations, i.e. $\varphi(a_1 \ldots a_n) = \varphi(a_{\sigma(1)} \ldots a_{\sigma(n)})$ for all $\sigma \in S_n$ and $n \in \mathbb{N}$. Are the free cumulants κ_n invariant under all permutations?