UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 6
Hand in on Wednesday, 28.11.18, before the lecture.

Exercise 1 (10 points).
Let b be a symmetric Bernoulli random variable; i.e. $b^2 = 1$ and its distribution (corresponding to the probability measure $\frac{1}{2}(\delta_{-1} + \delta_1)$) is given by the moments

$$\varphi(b^n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Show that the free cumulants of b are of the form

$$\kappa_n(b, b, \ldots, b) = \begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } n = 2k \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}$$

Exercise 2 (10 points).
Prove Theorem 3.24 from class; i.e., show that for random variables a_i $(i \in I)$ in some non-commutative probability space $(\mathcal{A}, \varphi)$ the following statements are equivaent.
(i) The random variables a_i $(i \in I)$ are freely independent.
(ii) All mixed cumulants in the random variables vanish, i.e., for any $n \geq 2$ and all $i(1), \ldots, i(n) \in I$ such that at least two of the indices $i(1), \ldots, i(n)$ are different we have:

$$\kappa_n(a_{i(1)}, a_{i(2)}, \ldots, a_{i(n)}) = 0.$$

Exercise 3 (10 points).
Let x and y be free random variables. Assume that x is even, i.e., all its odd moments vanish: $\varphi(x^{2k+1}) = 0$ for all $k \in \mathbb{N}_0$. Show that then also y and xyx are free.

Exercise 4.
In this exercise we want to adress the question of the existence of (nc) random variables for a prescribed distribution. Read lecture 6 (Pages 91-102) of the book "A. Nica, R. Speicher: Lectures on the Combinatorics of Free Probability" and do Exercise 6.16:

Let μ be a probability measure with compact support on $\mathbb{R}$. Show that one can find a $*$-probability space $(\mathcal{A}, \varphi)$ where is φ a faithful trace, and a sequence $(x_n)_{n\geq 0}$ of freely independent selfadjoint random variables in $\mathcal{A}$, such that each of the x_i's has distribution μ.