UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 7
Hand in on Wednesday, 12.12.18, before the lecture.

Exercise 1 (10 points).
1) Show that the Catalan numbers are exponentially bounded by

$$C_n \leq 4^n \qquad \text{for all } n \in \mathbb{N}.$$

2) Show that we have for the Möbius function on NC that

$$|\mu(0_n, \pi)| \leq 4^n \qquad \text{for all } n \in \mathbb{N} \text{ and all } \pi \in NC(n).$$

[Hint: use the multiplicativity of the Möbius function and the known value of the Möbius function $\mu(0_n, 1_n) = (-1)^{n-1}C_{n-1}$.]

Exercise 2 (10 points + 5 points*).
The complementation map $K : NC(n) \to NC(n)$ is defined as follows: We consider additional numbers $\bar{1}, \bar{2}, \ldots, \bar{n}$ and interlace them with $1, \ldots, n$ in the following alternating way:

$$1\,\bar{1}\,2\,\bar{2}\,3\,\bar{3}\,\ldots\,n\,\bar{n}.$$

Let $\pi \in NC(n)$ be a non-crossing partition of $\{1, \ldots, n\}$. Then its *Kreweras complement* $K(\pi) \in NC(\bar{1}, \bar{2}, \ldots, \bar{n}) \cong NC(n)$ is defined to be the biggest element among those $\sigma \in NC(\bar{1}, \bar{2}, \ldots, \bar{n})$ which have the property that $\pi \cup \sigma \in NC(1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n})$.

(i) Show that $K : NC(n) \to NC(n)$ is a bijection and a lattice anti-isomorphism, i.e., $\pi \leq \sigma$ implies $K(\pi) \geq K(\sigma)$.

(ii) As a consequence from (1) we have that

$$\mu(\pi, 1_n) = \mu(K(0_n), K(\pi)) \qquad \text{for } \pi \in NC(n).$$

Show from this that we have

$$|\mu(\pi, 1_n)| \leq 4^n \qquad \text{for all } n \in \mathbb{N} \text{ and all } \pi \in NC(n).$$

(iii)* Do we also have in general the estimate

$$|\mu(\sigma, \pi)| \leq 4^n \qquad \text{for all } n \in \mathbb{N} \text{ and all } \sigma, \pi \in NC(n) \text{ with } \sigma \leq \pi?$$

(iv) Show that we have for all $\pi \in NC(n)$ that

$$\#\pi + \#K(\pi) = n + 1.$$

Exercise 3 (10 points).
In the full Fock space setting of Exercise 3 from assignment 5 consider the operators $l_1 := l(u_1)$ and $l_2 := l(u_2)$ for two orthogonal unit vectors; according to part (ii) of that exercise l_1 and l_2 are $*$-free. Now we consider the two operators

$$a_1 := l_1^* + \sum_{n=0}^{\infty} \alpha_{n+1} l_1^n \qquad \text{and} \qquad a_2 := l_2^* + \sum_{n=0}^{\infty} \beta_{n+1} l_2^n,$$

for some sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ of complex numbers. (In order to avoid questions what we mean with this infinite sums in general, let us assume that the α_n and the β_n are such that the sum converges; for example, we could assume that $|\alpha_n|, |\beta_n| \leq r^n$ for some $0 < r < 1$. By the $*$-freeness of l_1 and l_2 we know that a_1 and a_2 are free. Prove now that $a_1 + a_2$ has the same moments as

$$a := l_1^* + \sum_{n=0}^{\infty} (\alpha_{n+1} + \beta_{n+1}) l_1^n.$$

Exercise 4 (10 points).
Let l be the creation operator from Exercise 3, Assignment 3 (or $l(v)$ from Exercise 3, Assignment 5) in the $*$-probability space $(B(\mathcal{H}), \varphi)$.

(i) Show that the only non-vanishing $*$-cumulants of l (i.e., all cumulants where the arguments are any mixture of l and l^*) are the second order cumulants, and for those we have:

$$\kappa_2(l, l) = \kappa_2(l^*, l^*) = \kappa_2(l, l^*) = 0, \qquad \kappa_2(l^*, l) = 1.$$

(ii) For a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of complex numbers we consider as in the previous exercise the operator

$$a := l^* + \sum_{n=0}^{\infty} \alpha_{n+1} l^n.$$

Show that the free cumulants of a are given by

$$\kappa_n^a = \kappa_n(a, a, \ldots, a) = \alpha_n.$$

(iii) Combine this with the previous exercise to show that the free cumulants are additive for free variables; i.e., for the a_1 and a_2 from the previous exercise we have for all $n \in \mathbb{N}$ that

$$\kappa_n^{a_1+a_2} = \kappa_n^{a_1} + \kappa_n^{a_2}.$$

[This is of course a consequence of the vanishing of mixed cumulants; however, the present proof avoids the use of the general cumulants functionals and was Voiculescu's original approach to the additivity of the free cumulants.]