UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
Prof. Dr. Roland Speicher
M.Sc. Felix Leid

Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 8
Hand in on Wednesday, 19.12.18, before the lecture.

Exercise 1 (10 points).
A free Poisson distribution μ_λ with parameter $\lambda > 0$ is defined in terms of cumulants by the fact that all free cumulants are equal to λ. Calculate from this the R-transform and the Cauchy transform of μ_λ. Use the latter to get via the Stieltjes inversion formula an explicit form for μ_λ.
[Note that for $\lambda < 1$ the distribution has atomic parts.]

Exercise 2 (10 points).
Let x_1 and x_2 be free symmetric Bernoulli variables; i.e., x_1 and x_2 are free, $x_1^2 = 1 = x_2^2$, and

$$\mu_{x_1} = \mu_{x_2} = \frac{1}{2}(\delta_{-1} + \delta_{+1}).$$

(i) Show directly, by using the definition of freeness, that the moments of $x_1 + x_2$ are given by

$$\varphi((x_1 + x_2)^n) = \begin{cases} 0, & n \text{ odd} \\ \binom{2m}{m}, & n = 2m \text{ even} \end{cases}$$

(ii) Show that the moments from (i) are the moments of the arcsine distribution, i.e., show that

$$\frac{1}{\pi}\int_{-2}^{2} t^{2m} \frac{1}{\sqrt{4-t^2}} dt = \binom{2m}{m}.$$

Exercise 3 (10 points).

(i) Let $\lambda > 0$ and ν be a probability measure on $\mathbb{R}$ with compact support. Show that the limit in distribution for $N \to \infty$ of

$$[(1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\nu]^{\boxplus N}$$

has free cumulants $(\kappa_n)_{n \geq 1}$, which are given by

$$\kappa_n = \lambda \cdot m_n(\nu), \qquad \text{where} \qquad m_n(\nu) = \int t^n d\nu(t)$$

is the n-th moment of ν.

[A distribution with those cumulants is called a *free compound Poisson distribution* (with rate λ and jumb distribution ν).]

(ii) Let s and a be two free selfadjoint random variables in some $*$- probability space $(\mathcal{A}, \varphi)$ such that s is a standard semicircular element and a has distribution $\mu_a = \nu$. Show the following. The free cumulants of sas are given by

$$\kappa_n(sas, sas, \ldots, sas) = \varphi(a^n) \qquad \text{for all } n \geq 1,$$

i.e., sas is a compound Poisson element with rate $\lambda = 1$ and jump distribution ν.