UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Free Probability
Winter term 2018/19

Assignment 9
Hand in on Wednesday, 09.01.19, before the lecture.

Exercise 1 (10 points)**.**
In the proof of Theorem 5.11 we used the following consequence of the Schwarz Lemma (alternatively, one can address this also as the simple part of the Denjoy-Wolff Theorem). Suppose $f : \mathbb{D} \to \mathbb{D}$ is a non-constant holomorphic function on the unit disc

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$$

and it is not an automorphism of $\mathbb{D}$ (i.e., not of the form $\lambda(z-\alpha)/(1-\bar{\alpha}z)$ for some $\alpha \in \mathbb{D}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$). If there is a $z_0 \in \mathbb{D}$ with $f(z_0) = z_0$, then for all $z \in \mathbb{D}$, $f^{\circ n}(z) \to z_0$. In particular, the fixed point is unique.
Prove this by an application of the Schwarz Lemma.

Exercise 2 (10 points)**.**
Let μ be the standard semicircular distribution (i.e., $R_\mu(z) = z$) and ν be the free Poisson distribution of parameter 1 (i.e., $R_\nu(z) = 1/(1-z)$). Calculate (explicitly or numerically) the distribution $\mu \boxplus \nu$, by producing plots for its density, via

(i) determining its Cauchy transform from its R-transform:

$$R(z) = z + \frac{1}{1-z}.$$

(ii) determining its Cauchy transform G from the subordination equation:

$$G(z) = G_\nu(z - G(z)).$$

Exercise 3 (20 points)**.**
A probability measure μ on $\mathbb{R}$ is called *infinitely divisible (in the free sense)* if, for each $N \in \mathbb{N}$, there exists a probability measure μ_N on $\mathbb{R}$ such that

$$\mu = \mu_N^{\boxplus N}.$$

(This is equivalent to requiring that the free convolution semigroup $\{\mu^{\boxplus t} \mid t \geq 1\}$ can be extended to all $t \geq 0$; the μ_N from above are then $\mu^{\boxplus 1/N}$.)

(i) Show that a free compound Poisson distribution (which was defined on Assignment 8, Exercise 3) is infinitely divisible.

(ii) Show the the R-transform of a free compound Poisson distribution with rate λ and jump distribution ν is given by

$$R(z) = \lambda \int \frac{t}{1 - tz} d\nu(t),$$

and thus can be extended as an analytic function to all of $\mathbb{C}^-$.

(iii) Show that a semicircular distribution is infinitely divisible.

(iv) Show that a semicircular distribution can be approximated in distribution by free compound Poisson distributions.

[One has that any infinitely divisible distribution can be approximated by free compound Poisson distributions. Furthermore, infinitely divisible distributions are characterized by the fact that their R-transforms have an analytic extension to $\mathbb{C}^-$.]