Hardy spaces   Assignment 1
Due Thursday, December 19, at the beginning of class

Question 1 (4 points)

(a) (1 point) Show that if $f, g \in H^2$, then $f \cdot g \in H^1$ with $\|f \cdot g\|_1 \leq \|f\|_2 \|g\|_2$.

(b) (3 points) In class, we showed for all $f \in H^1$ and all $z \in \mathbb{D}$ the estimate

$$|f(z)| \leq \frac{1}{1-|z|} \|f\|_1.$$

Use part (a) to show that this estimate is sharp up for all $z \in \mathbb{D}$ up to a multiplicative constant. That is, show that there exists $c > 0$ so that for all $z \in \mathbb{D}$, there exists $f \in H^1$ with

$$|f(z)| \geq \frac{c}{1-|z|} \|f\|_1.$$

Question 2 (4 points)

Let $a \in \mathbb{D}$ and let

$$\varphi_a : \overline{\mathbb{D}} \to \mathbb{C}, \quad z \mapsto \frac{a-z}{1-\overline{a}z}.$$

Show that φ_a maps $\mathbb{D}$ bijectively onto $\mathbb{D}$ and $\mathbb{T}$ bijectively onto $\mathbb{T}$. Moreover, show that $\varphi_a(\varphi_a(z)) = z$ for all $z \in \overline{\mathbb{D}}$.
(*Hint: You may save some calculations by first computing* $1 - |\varphi_a(z)|^2$.)

Question 3 (5 points)

Let $\alpha > 0$ and let

$$f : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{1}{(1-z)^\alpha}.$$

Show that $f \in H^p$ if and only if $p < \frac{1}{\alpha}$. Thus, conclude that $H^p \subsetneq H^q$ if $q < p$.
(*Possible strategy: First show that* $f_\alpha \in H^p$ *if and only if the function on* $\mathbb{T}$ *defined by the same formula belongs to* $L^p(\mathbb{T})$. *Then approximate* $|1-e^{it}|^2$ *in a neighborhood of* $t = 0$ *by a Taylor polynomial.*)

Question 4 (7 points)

Let $1 \leq p < \infty$, let A denote the two dimensional Lebesgue measure and let

$$L^p_a(\mathbb{D}) = \Big\{ f \in \mathcal{O}(\mathbb{D}) : \|f\|_{L^p(\mathbb{D})} := \Big(\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p \, dA(z) \Big)^{1/p} < \infty \Big\}.$$

(a) (1 point) Suppose that $1 \leq p \leq q < \infty$. Show that $L^q_a(\mathbb{D}) \subseteq L^p_a(\mathbb{D})$ and that $\|f\|_{L^p(\mathbb{D})} \leq \|f\|_{L^q(\mathbb{D})}$ for all $f \in L^q_a(\mathbb{D})$.

(b) (3 points) Show that if $f \in L^1_a(\mathbb{D})$ and $z \in \mathbb{D}$, then

$$|f(z)| \leq \frac{1}{(1-|z|)^2} \|f\|_{L^1(\mathbb{D})}.$$

(c) (3 points) Conclude that $L^p_a(\mathbb{D})$ is a closed subspace of $L^p(\mathbb{D})$ and hence a Banach space.