Hardy spaces Assignment 2

Due Thursday, January 9, at the beginning of class

Question 1 (5 points)

(a) (3 points) Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on $\mathbb{D}$ with reproducing kernel K. Let $(e_n)_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Show that

$$K(z,w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$$

for all $z, w \in \mathbb{D}$.

(b) (2 points) Use part (a) to compute the reproducing kernel of H^2 in yet another way.

Question 2 (5 points)

Let (s_n) be a sequence in a normed space and let

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} s_n$$

be the Cesàro means.

(a) (2 points) Suppose that $\lim_{n\to\infty} s_n = s$. Prove that $\lim_{N\to\infty} \sigma_N = s$.

(b) (3 points) Suppose now that $s_n = \sum_{k=0}^{n} a_k$ for some sequence $(a_k)_{k=0}^{\infty}$. Suppose further that $\lim_{k\to\infty} k a_k = 0$. Prove that if (σ_N) converges, then so does (s_n). (*Hint: Compute an expression for $S_N - \sigma_N$ in terms of a_n.*)

Question 3 (5 points)

Let $f \in L^1(\mathbb{T})$.

(a) (2 points) Show that $|\widehat{f}(n)| \leq \|f\|_{L^1(\mathbb{T})}$ for all $n \in \mathbb{Z}$.

(b) (3 points) Show that $\lim_{n\to\infty} \widehat{f}(n) = 0$ and $\lim_{n\to-\infty} \widehat{f}(n) = 0$. (*Hint: First show the result for $f \in L^2(\mathbb{T})$. Then use density of $L^2(\mathbb{T})$ in $L^1(\mathbb{T})$.*)

Question 4 (5 points)

Let $f \in L^\infty(\mathbb{T})$ with $\|f\|_{L^\infty(\mathbb{T})} = 1$.

(a) (2 points) Show that $\|\sigma_N[f]\|_{L^\infty(\mathbb{T})} \leq 1$ for all $N \in \mathbb{N}$.

(b) (3 points) Suppose that there exists $z \in \mathbb{T}$ and $N \in \mathbb{N}$ with $|\sigma_N[f](z)| = 1$. Show that f is constant almost everywhere.