Hardy spaces Assignment 3
Due Thursday, January 16, at the beginning of class

Question 1 (3 points)
Let δ_1 denote the Dirac measue at 1.

(a) (1 point) Compute the Fourier coefficients of δ_1.

(b) (2 points) Compute the Fejér means $\sigma_N[\delta_1]$. What does the theorem about weak-* convergence of the Fejér means say in this case?

Question 2 (5 points)
Let $1 \leq p < \infty$ and let p' be the conjugate exponent of p. Prove the following special case of the duality between $L^p(\mathbb{T})$ and $L^{p'}(\mathbb{T})$, which we used in class: If $g : \mathbb{T} \to \mathbb{C}$ is a measurable function, then

$$\|g\|_p = \sup \Big\{ \int_{\mathbb{T}} |gh| \, dm : h \in L^{p'}(\mathbb{T}), \|h\|_{p'} \leq 1 \Big\}.$$

Question 3 (5 points)
Let $u : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function that is harmonic on $\mathbb{D}$. Show that if there exists $z_0 \in \mathbb{D}$ with

$$|u(z_0)| = \sup_{z \in \overline{\mathbb{D}}} |u(z)|,$$

then u is constant. Deduce that

$$\sup_{z \in \overline{\mathbb{D}}} |u(z)| = \sup_{z \in \mathbb{T}} |u(z)|.$$

Question 4 (7 points)
For $n \in \mathbb{N}$, let

$$L_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} c_n(1-x^2)^n, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $c_n \in \mathbb{R}$ is chosen so that $\int_{-\infty}^{\infty} L_n(x) \, dx = 1$.

(a) (2 points) Show that (L_n) is an approximate identity for $\mathbb{R}$, that is, show that

  (i) $L_n \geq 0$ for all $n \in \mathbb{N}$,

  (ii) $\int_{-\infty}^{\infty} L_n(x) \, dx = 1$ for all $n \in \mathbb{N}$,

  (iii) for all $\delta > 0$, we have $\lim_{n\to\infty} \sup_{x \in \mathbb{R}, |x| \geq \delta} |L_n(x)| = 0$.

(b) (1 point) Let $f : [0,1] \to \mathbb{C}$ be continuous and let

$$p_n(x) = (f * L_n)(x) = \int_0^1 f(t) L_n(x-t) \, dt.$$

Show that p_n is a polynomial for all $n \in \mathbb{N}$.

(c) (3 points) Let $f : [0,1] \to \mathbb{C}$ be continuous with $f(0) = f(1) = 0$. Show that (p_n) converges to f uniformly on $[0,1]$.

(d) (1 point) Conclude that the polynomials are dense in the space $C[0,1]$ of continuous functions with respect to the supremum norm on $[0,1]$.