Hardy spaces Assignment 4

Due Thursday, January 23, at the beginning of class

Question 1 (5 points)

Let $1 \leq p \leq \infty$. For a harmonic function $u : \mathbb{D} \to \mathbb{C}$, let

$$\|u\|_p = \sup_{0 \leq r < 1} M_p(r, u)$$

and let

$$h^p = \{u : \mathbb{D} \to \mathbb{C} : u \text{ is harmonic and } \|u\|_p < \infty\},$$

equipped with the norm $\| \cdot \|_p$. Show that the map

$$\Phi : L^p(\mathbb{T}) \to h^p, \quad f \mapsto P[f],$$

is an isometry. Moreover, show that Φ is an isomorphism if $1 < p \leq \infty$ and that Φ is not surjective if $p = 1$.

Question 2 (4 points)

Let $u : \mathbb{D} \to \mathbb{R}$ be harmonic. Show that $\sup_{0 \leq r < 1} M_1(r, u) < \infty$ if and only if u is the difference of two non-negative harmonic functions on $\mathbb{D}$.

Question 3 (5 points)

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $\operatorname{Re} f(z) \geq 0$ for all $z \in \mathbb{D}$ and $f(0) = 1$. Prove that there exists a positive probability measure $\mu \in M(\mathbb{T})$ with

$$f(z) = \int_{\mathbb{T}} \frac{w + z}{w - z} \, d\mu(w)$$

for all $z \in \mathbb{D}$.

Question 4 (6 points)

Let $u : \mathbb{D} \to \mathbb{R}$ be harmonic. For $1 \leq p \leq \infty$ let

$$h^p_{\mathbb{R}} = \{u : \mathbb{D} \to \mathbb{R}, u \in h^p\},$$

equipped with $\| \cdot \|_p$ (see Question 1).

(a) (2 points) Show that there exists a unique harmonic function $v : \mathbb{D} \to \mathbb{R}$ so that $u + iv$ is holomorphic and $v(0) = 0$. We will write $v = H[u]$.

(b) (2 points) Show that $H : h^2_{\mathbb{R}} \to h^2_{\mathbb{R}}$ is linear and bounded. (*Hint: It might be convenient to use Question 1.*)

(c) (2 points) Show that H does not define a bounded linear map from $h^\infty_{\mathbb{R}}$ to $h^\infty_{\mathbb{R}}$. (*Hint: Consider $f(z) = \log(1 - z)$.*)