Hardy spaces Assignment 5
Due Thursday, January 30, at the beginning of class

Question 1 (5 points)
Let $w \in L^1(\mathbb{T})$ be a non-negative function. Use the Szegő–Kolmogoroff–Krein theorem (and related results) to show that if $\log w \in L^1(\mathbb{T})$, then there exists $f \in H^2(\mathbb{T})$ with $w = |f|^2$ almost everywhere.

Question 2 (5 points)
Let $\mu \in M(\mathbb{T})$ be a positive measure and let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$. Recall that

$$d(\mu) = \inf_{p \in P_0} \int_{\mathbb{T}} |1 - p|^2 \, d\mu,$$

where P_0 is the space of all polynomials vanishing at 0. Show that the following assertions are equivalent:

(i) $d(\mu) > 0$.

(ii) $H^2(\mu) \neq L^2(\mu)$.

(iii) There is a constant $C \geq 0$ so that

$$|p(0)| \leq C\|p\|_{L^2(\mu)}$$

for all polynomials p.

Question 3 (5 points)
Let $g \in H^\infty(\mathbb{D})$. Show that the following assertions are equivalent:

(i) g is inner.

(ii) $\|gf\|_2 = \|f\|_2$ for all $f \in H^2(\mathbb{D})$.

(iii) $\|g\|_2 = \|g\|_\infty = 1$.

Question 4 (5 points)
Let $U \subset \mathbb{C}$ be open and let (f_n) be a sequence of holomorphic functions on U such that $\sum_{n=0}^{\infty}(f_n - 1)$ converges absolutely and uniformly on compact subsets of U. Show that

$$f := \lim_{N \to \infty} \prod_{n=1}^{N} f_n$$

converges uniformly on compact subsets of U. Moreover, show that if $f(a) = 0$, then $f_n(a) = 0$ for at most finitely many f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeroes of the f_n at a. (*Hint: Use logarithms to convert products into sums.*)