K-Theorie für C^*-Algebren

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Einleitung

„K-Theorie hat das Studium von Operatoralgebren revolutioniert.“

Bruce Blackadar

Es gibt eine topologische K-Theory (Atiyah-Hirzebruch, 60er Jahre des letzten Jahrhunderts); diese verfolgt den Ansatz, Vektorbündel mit algebraischen Methoden zu untersuchen. Daraus entwickelte sich die K-Theorie für C^*-Algebren in den 70er Jahren des letzten Jahrhunderts. Parallel entwickelt haben sich die Ext-Theorie und die K-Homologie-Theorie, die zusammen mit der K-Theorie von Kasparov in den 80ern in die KK-Theorie zusammengeführt wurden. Außerdem gibt es noch die E-Theorie (Connes-Higson, 90er Jahre des letzten Jahrhunderts), das ist bivariante oder auch equivariante K-Theorie mit Gruppenwirkungen. Baum-Connes-Vermutung.

Die Idee, die der K-Theorie zugrunde liegt, ist

$$\{C^*\text{-Algebren}\} \overset{K_1}{\underset{K_0}{\rightrightarrows}} \{\text{Abelsche Gruppen}\},$$

der Funktor K_0 „zählt Projektionen“, der Funktor K_1 „zählt Unitäre“. Ist $A \cong B$, so ist $K_i(A) = K_i(B)$ für $i \in \{0,1\}$.

Das Elliot-Programm stellt sich die Frage, wann man „zurückgehen kann“, beziehungsweise wann man eine Klasse $\mathfrak{C} \subseteq \{C^*\text{-Algebren}\}$ mit der Eigenschaft, dass wenn $K_iA \cong K_iB$ auch gilt $A \cong B$ für $A, B \in \mathfrak{C}$.

Klassifikationsresultate im Rahmen des Elliot-Programms sind:

Satz .1 (Tikcaisis, White, Winter): *Seien A, B separable, unitale, einfache, unendlich-dimensionale C^*-Algebren mit endlicher nuklearer Dimension, die die universelle Koeffizienteneigenschaft haben. Dann gilt $A \cong B$ genau dann wenn* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Satz .2 (Eilers, Restorff, Ruiz, Sørensen): *Seien Γ_1 und Γ_2 endliche Graphen. Es gilt $C^*(\Gamma_1) = C^*(\Gamma_2)$ genau dann, wenn* $\mathrm{Ell}'(C^*\Gamma^1) \cong \mathrm{Ell}'(C^*\Gamma_2)$.

Für von Neumann-Algebren entspricht K-Theorie in etwa der Typenklassifizierung; also mehr oder weniger „nutzlos“.

Schöne Eigenschaften der K-Theorie sind:

- Funktorialität: Ist $\varphi\colon A \to B$ ein *-Homomorphismus, dann ist $\varphi_*\colon K_i(A) \to K_i(B)$ für $i \in \{0,1\}$ ein Gruppenhomomorphismus; es ist $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$, ...

- Homotopieinvarianz: Sind $\varphi, \psi\colon A \to B$ homotope *-Homomorphismen, dann ist $\varphi_* = \psi_*$; ist $A \sim_h B$, dann ist $K_iA \cong K_iB$.

- Stabilität: Es ist $K_i(M_n(A)) = K_i(A)$ und $K_i(A \otimes K(H)) = K_i(A)$, wenn H ein separabeler Hilbertraum ist,

- Bott-Periodizität: Man kann für $n \in \mathbb{N}$ auf natürliche Weise $K_n(A)$ definieren und zeigen, dass $K_{n+2}(A) \cong K_n$. Es gibt also nur K_0 und K_1.

- Ausschneidung: Ist

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\varphi} B \longrightarrow 0$$

eine kurze exakte Sequenz, d. h. ι ist injektiv, φ ist surjektiv und $\iota(I) = \ker\varphi$, also $B \cong A/I$ und $I \lhd A$, dann ist

$$\begin{array}{ccccc} K_0I & \longrightarrow & K_0A & \longrightarrow & K_0B \\ \uparrow & & & & \downarrow \\ K_1B & \longleftarrow & K_1A & \longleftarrow & K_1I \end{array}$$

exakt und gibt es $\psi\colon B \to A$ mit $\varphi \circ \psi = \mathrm{id}_B$, so heißt die Sequenz *split exakt*. In diesem Fall ist $K_iA = K_iB \oplus K_iI$.

- Korrespondenz zum Klassischen: Es gilt $K_n(C_0(X)) \cong K^n(X)$.

Ziele für die Vorlesung sind:

(a) Definiere K_0 und K_1, weise obige Eigenschaften nach,

(b) Konkrete Berechnungen von K-Theorien von bestimmten C^*-Algebren (oft benötigt man dafür lediglich *Eigenschaften*, nicht die Definition von K_0 und K_1),

(c) Blumige Bemerkungen.

Kapitel I.

Äquivalenzrelationen auf Unitären

Definition I.1: Seien G eine topologische Gruppe und g_1, g_2 Elemente von G. Wir sagen, g_1 und g_2 seien *homotop*, in Zeichen $g_1 \sim_h g_2$, falls es einen stetigen Pfad $\gamma\colon [0,1] \to G$ mit $\gamma(0) = g_1$, $\gamma(1) = g_2$ gibt.

Wir nennen G_0 die Zusammenhangskomponente des neutralen Elements von G, also $G_0 := \{g \in G \mid g \sim_h e\} \subseteq G$.

Lemma I.2: *Homotopie ist eine Äquivalenzrelation auf G und $G_0 \subseteq G$ ist eine Untergruppe.*

Beweis: Ist $g \in G_0$, dann ist g^{-1} homotop zum neutralen Element per $(\gamma_t^{-1})_{t\in[0,1]}$, d. h. $g^{-1} \in G_0$. Sind g_1 und g_2 homotop zum neutralen Element via $(\gamma_t^1)_{t\in[0,1]}$ beziehungsweise $(\gamma_t^2)_{t\in[0,1]}$, dann auch $g_1 g_2$ per $(\gamma_t^1 \gamma_t^2)_{t\in[0,1]}$. □

Definition I.3: Sei A eine unitale C^*-Algebra. Dann ist

$$U(A) := \{u \in A \text{ unitär}\}$$

eine topologische Gruppe mit neutralem Element 1 mit Zusammenhangskomponente des neutralen Elements $U_0(A) := \{u \in U(A) \mid u \sim_h 1\}$.

Lemma I.4: *Seien A eine unitale C^*-Algebra und $u \in U(A)$.*

(i) *Gilt $\|1 - u\| < 2$, dann gibt es ein selbstadjungiertes $h \in A$ mit $u = e^{ih} \sim_h 1$, d. h. $u \in U_0(A)$,*

(ii) *Es ist $U_0(A) = \{e^{ih_1} \cdots e^{ih_n} \mid h_1, \ldots, h_n \in A \text{ selbstadjungiert}, n \in \mathbb{N}\}$,*

(iii) *Ist B eine weitere unitale C^*-Algebra und $\sigma\colon A \to B$ ein unitaler surjektiver *-Homomorphismus, dann gilt $\sigma(U_0(A)) = U_0(B)$.*

Beweis: (i) Es gilt $-1 \notin \operatorname{Sp} u \subseteq \mathbb{S}^1$, da $\|1-u\| < 2$ (Funktionalkalkül). Die Funktion $e^{\mathrm{i}\cdot}\colon [0,2\pi) - \{\pi\} \to \mathbb{S}^1 - \{1\}$ ist invertierbar mit stetiger Umkehrfunktion arg. Wegen des Funktionalkalküls ist $\arg\colon \operatorname{Sp} u \to A$ eine wohldefinierte reell-wertige Funktion, d. h. $h := \arg(u)$ is selbstadjungiert. Es gilt $e^{\mathrm{i}h} = e^{\mathrm{i}\arg(u)} = u$.

Für die Homotopie setzen wir $\gamma_t := e^{\mathrm{i}th}$ für $t \in [0,1]$; es ist also $\gamma(0) = 1$ und $\gamma(1) = u$

(ii) „$\supseteq$“: Mit $\gamma_t := e^{\mathrm{i}th_1} \cdots e^{\mathrm{i}th_n}$ ist $e^{\mathrm{i}h_1} \cdots e^{\mathrm{i}h_n} \sim_h 1$.

„$\subseteq$“: Für $u \sim_h 1$ via $(\gamma_t)_{t\in[0,1]}$ wähle eine Zerlegung $0 = t_0 < \cdots < t_n = 1$ mit $\|\gamma(t_{i+1}) - \gamma(t_i)\| < 2$ (so eine Zerlegung gibt es wegen der Stetigkeit von $t \mapsto \gamma_t$). Dann gilt

$$u = \gamma(1) = \gamma(t_n)\gamma(t_{n-1})^{-1}\gamma(t_{n-1})\gamma(t_{n-2})^{-1}\gamma(t_{n-2}) \cdots \gamma(t_1)\gamma(t_0)^{-1}\gamma(t_0),$$

wobei $\gamma(t_{j+1})\gamma(t_j)^{-1} = \exp^{\mathrm{i}h_j}$, denn

$$\|1 - \gamma(t_i)\gamma(t_{i-1})^{-1}\| = \|(\gamma(t_{i-1}) - \gamma(t_i))\gamma(t_i)^{-1}\| \leq \|\gamma(t_{i-1}) - \gamma(t_i)\| < 2.$$

(iii) „$\subseteq$“: Es ist $\sigma(e^{\mathrm{i}h_1} \cdots e^{\mathrm{i}h_n}) = e^{\mathrm{i}\sigma(h_1)} \cdots e^{\mathrm{i}\sigma(h_n)} \in U_0$ nach (ii).

„$\supseteq$“: Sei $v := e^{\mathrm{i}h_1} \cdots e^{\mathrm{i}h_n} \in U_0 B$. Da σ surjektiv ist, existieren $h_1', \ldots, h_n' \in A$ mit $\sigma(h_i') = h_i$. Setze $g_i := \frac{1}{2}(h_i' + h_i'^*)$. Es gilt weiter $\sigma(g_i) = h_i$ und für $u := e^{\mathrm{i}g_1} \cdots e^{\mathrm{i}g_n}$ ist $\sigma(u) = v$. □

Lemma I.5: (i) *Es ist $U_0(M_n(\mathbb{C})) = U(M_n(\mathbb{C}))$.*

(ii) *Ist A eine unitale C^*-Algebra, so ist $U_0(A) \subseteq U(A)$ eine normale Untergruppe,*

(iii) *Sind $u, v \in U(A)$ mit $\|u - v\| < 2$, dann sind u und v homotop,*

(iv) *Die Zusammenhangskomponente $U_0(A)$ des neutralen Elements ist offen und abgeschlossen in $U(A)$.*

Beweis: (i) Für $u \in M_n(\mathbb{C})$ ist $\operatorname{Sp} u$ endlich, also ist ein Zweig des Logarithmus stetig (siehe Beweis von Lemma 1.4).

(ii) Sei $u \in U_0(A)$ mit der zugehörigen $(\gamma_t)_{t\in[0,1]}$, weiter sei $v \in U(A)$. Dann ist vuv^* homotop zu 1 vermöge $(v\gamma_t v^*)_{t\in[0,1]}$, also ist $vuv^* \in U_0(A)$ und $U_0(A)$ ist normal in $U(A)$.

(iii) Es gilt $\|1 - u^*v\| = \|u^*(u - v)\| \leq \|u - v\| < 2$, nach (Lemma I.4) ist also u^*v homotop zu 1 und damit ist $v = u(u^*v)$ homotop zu u.

(iv) Zur Offenheit: Sind $u \in U_0(A)$ und $v \in U(A)$ mit $\|u - v\| < 2$, dann liefert (iii) dass $v \sim_h u \sim_h 1$, d. h. $B(u, 2) \subseteq U_0(A)$ für alle $u \in U$. Zur Abgeschlossenheit: Sei $(u_n)_{n\in\mathbb{N}}$ eine Folge in $U_0(A)$ mit Grenzwert $u \in U$. Dann gibt es eine natürliche Zahl N mit $\|u_N - u\| < 2$, nach (iii) also $u \sim_h u_N \sim_h 1$, d. h. $u \in U_0(A)$. □

Lemma I.6: *Sei A eine unitale C^*-Algebra. Dann gilt in $U(M_2(A))$:*

$$\begin{pmatrix} u & \\ & v \end{pmatrix} \sim_h \begin{pmatrix} uv & \\ & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & \\ & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & \\ & u \end{pmatrix}.$$

Insbesondere gilt $\left(\begin{smallmatrix} u & \\ & u^* \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$.

Beweis: Für $t \in [0, 1]$, setze

$$\gamma_t := \begin{pmatrix} \cos(t\frac{\pi}{2}) & \sin(t\frac{\pi}{2}) \\ -\sin(\pi\frac{t}{2}) & \cos(t\frac{\pi}{2}) \end{pmatrix}$$

Es gilt $\gamma_t \in U(M_2(A))$, da

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}^* \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2 + s^2 & 0 \\ 0 & c^2 + s^2 \end{pmatrix}.$$

Nun ist

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \gamma_1 \sim_h \gamma_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad \begin{pmatrix} u & \\ & v \end{pmatrix} \sim_h \begin{pmatrix} -u & \\ & -v \end{pmatrix}$$

via $\gamma'_t := e^{\mathrm{i}t\pi}\left(\begin{smallmatrix} u & \\ & v \end{smallmatrix}\right)$, damit ist

$$\begin{pmatrix} u & \\ & v \end{pmatrix} \sim_h \begin{pmatrix} -u & \\ -v & \end{pmatrix} = \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} v & \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \sim_h \begin{pmatrix} uv & \\ & 1 \end{pmatrix}$$ □

Kapitel II.

Äquivalenzrelationen für Projektionen

Definition II.1: Seien A eine C^*-Algebra und $p, q \in A$ seien Projektionen (d. h. $p = p^2 = p^*$, $q = q^2 = q^*$).

(i) Die Projektionen p und q heißen Murray-von Neumann-äquivalent, in Zeichen $p \sim q$, falls es ein $v \in A$ mit $v^*v = p$, $vv^* = q$ gibt.

(ii) Die Projektionen p und q heißen Unitär-äquivalent, in Zeichen $p \sim_u q$, falls es $u \in U(A)$ mit $upu^* = q$ gibt,

(iii) Die Projektionen p und q heißen U_0-äquivalent,in Zeichen $p \sim_{U_0} q$, falls es $u \in U_0(A)$ mit $upu^* = q$ gibt,

(iv) Die Projektionen p und q heißen homotop, in Zeichen $p \sim_h q$, falls es einen stetigen Weg $\gamma\colon [0,1] \to A$ gibt, sodass $\gamma(0) = p$, $\gamma(1) = q$ und $\gamma(t)$ eine Projektion für jedes $t \in [0,1]$ ist.

Bemerkung II.2: Ein Element $v \in A$ ist eine partielle Isometrie (d. h. $v = vv^*v$) genau dann, wenn v^*v eine Projektion ist, was genau dann gilt, wenn vv^* eine Projektion ist.

Für v mit $v = vv^*v$ gilt $(v^*v)(v^*v) = v^*v$. Ist v^*v eine Projektion, so ist

$$\|v - vv^*v\|^2 = \|(v - vv^*v)(v - vv^*v)\| = \|v^*v - v^*vv^*v - v^*vv^*v + v^*vv^*vv^*v\| = 0.$$

Ist $v \in B(H)$ eine partielle Isometrie, dann ist $v\colon (v^*v)H \to (vv^*)H$ eine Isometrie.

Beispiel II.3: Ist H ein Hilbertraum mit $\dim H < \infty$, so ist $A = M_n(\mathbb{C}) = B(H)$. Zwei Projektionen $p, q \in A$ sind Murray-von Neumann-äquivalent genau dann, wenn $pH \cong qH$; also wenn $\operatorname{rank} p = \operatorname{rank} q$.

Lemma II.4: *Die Äquivalenzbegriffe für Projektionen aus Definition II.1 sind in der Tat Äquivalenzrelationen.*

Beweis: Für die Transitivität: Seien p, q und r Projektionen in A mit $p \sim q \sim r$. Dann sind $p = v^*v$, $q = vv^* = w^*w = ww^*$. Mit $z := wv$ ist

$$p = v^*(vv^*)v = z^*z \sim zz^* ... \qquad \square$$

Lemma II.5: *Sei A eine unitale C^*-Algebra, p und q seien Projektionen in A. Ist $\|p - q\| < 1$, dann ist $p \sim_{u_0} q$.*

Beweis: Setze $x := 2p-1$, $y := 2q-1$. Dann gelten $x = x^*$, $y = y^*$, $x^2 = y^2 = 1$, d. h. x und y sind Symmetrien, insbesondere also Unitäre. Damit ist $xy \in U(A)$ und es gilt

$$\|1 - xy\| = \|x(x - y)\| \leq \|x - y\| = 2\|p - q\| < 2.$$

Nach (Lemma I.4) (i) gibt es ein selbstadjungiertes $h \in A$, sodass $xy = e^{\mathrm{i}h}$ und $yx = (xy)^* = e^{-\mathrm{i}h}$.

Die Abbildung

$$\alpha\colon A \longrightarrow A, \qquad a \longmapsto xax^*$$

ist ein *-Homomorphismus, da x unitär ist. Des Weiteren ist $\arg(xy) = h$, $\arg(yx) = -h$, wobei arg die Funktion aus dem Beweis von (Lemma I.4) (a) ist. Da gilt $\alpha(xy) = x^2yx = yx$, ist $\alpha(h) = \alpha(\arg(xy)) = \arg(\alpha(xy)) = \arg(yx) = -h$. Setze jetzt $u := e^{-(\mathrm{i}/2)h} \in U_0(A)$. Für die Funktion g definiert durch $g(t) := e^{-(\mathrm{i}/2)t}$ haben wir $\alpha(u) = \alpha(g(h)) = g(\alpha(h)) = g(-h) = e^{-(\mathrm{i}/2)h}$, also ist $uxu^* = x(xux)u^* = x\alpha(u)u^* = xe^{(\mathrm{i}/2)h}e^{(\mathrm{i}/2)h} = xxy = y$ was

$$upu^* = u\left(\frac{x+1}{2}\right)u^* = \frac{y+1}{2} = q. \qquad \square$$

Proposition II.6: *Sei A eine unitale C^*-Algebra und p, q seien Projektionen in A. Dann gilt*

$$p \sim_h q \Longleftrightarrow p \sim_{U_0} q \Longrightarrow p \sim_u q \Longrightarrow p \sim q.$$

Beweis: „$p \sim_h q \Rightarrow p \sim_{U_0} q$“: Sei $p \sim_h q$ durch die Homotopie $(\gamma_t)_{t\in[0,1]}$. Wegen der Stetigkeit von $t \mapsto \gamma_t$ finden wir eine Zerlegung $0 = t_0 < \cdots < t_n = 1$, sodass $\|\gamma(t_{i-1}) - \gamma(t_i)\| < 1$. Wegen (Lemma II.5) gibt es $u_i \in U_0(A)$, sodass $u_i\gamma(t_{i-1})u_i^* = \gamma(t_i)$. Setze $u := u_1 \cdots u_n \in U_0(A)$. Dann ist

$$upu^* = u_n \cdots u_1\gamma(t_0)u_1^* \cdots u_n^* = \gamma(t_n) = q.$$

„$p \sim_{U_0} q \Rightarrow p \sim_h q$": Für $u \in U_0(A)$ mit $upu^* = q$ haben wir eine Familie $(\gamma_t)_{t\in[0,1]}$ in $U_0(A)$ mit $\gamma_0 = u$, $\gamma_1 = 1$. Via $\gamma_t' := \gamma_t p \gamma_t^*$ ist nun $p \sim_h q$.

„$p \sim_{U_0} q \Rightarrow p \sim_U q$": Das ist klar, da $U_0(A)$ eine Teilmenge von $U(A)$ ist.

„$p \sim_u q \Rightarrow p \sim q$": Sei $u \in U(A)$ mit $upu^* = q$. Das Element $v := up$ leistet $v^*v = p$ und $vv^* = q$, was wir zeigen wollten. □

Proposition II.7: *Betrachte $A \cong \left(\begin{smallmatrix} A & \\ & 0 \end{smallmatrix}\right) \subseteq M_2(A)$ als C^*-Unteralgebra. Ferner seien p und q Projektionen in A. Dann gilt $p \sim_h q$ in $M_2(A)$ genau dann, wenn $p \sim q$ in A.*

Beweis: „$\Rightarrow$": Nach (Proposition II.6) gibt es $u \in U_0(M_2(A'))$ mit $upu^* = q$, wobei A' eine Unitalisierung von A ist. Setzen wir $v := up$ so ist

$$v = up = (upu^*)up = qup = \begin{pmatrix} q & \\ & 0 \end{pmatrix} u \begin{pmatrix} p & \\ & 0 \end{pmatrix} \in \begin{pmatrix} A & \\ & 0 \end{pmatrix} \subseteq M_2(A) \subseteq M_2(A')$$

und $v^*v = p$, $vv^* = q$.

„$\Leftarrow$": Sei $v \in A$ mit $v^*v = p$ und $vv^* = q$. Dann ist

$$u := \begin{pmatrix} v & -(1-q) \\ 1-p & v^* \end{pmatrix}$$

das gesuchte Unitäre, denn

$$\begin{aligned} uu^* &= \begin{pmatrix} v & -(1-q) \\ 1-p & v^* \end{pmatrix} \cdot \begin{pmatrix} v^* & 1-p \\ -(1-q) & v \end{pmatrix} \\ &= \begin{pmatrix} vv^* + (1-q) & v(1-p) - (1-q)v \\ (1-p)v^* - v^*(1-q) & vv^* + (1-p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Wegen $u \sim_h I_2$ wissen wir

$$u \sim_h u \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \sim_h \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

durch

$$u \begin{pmatrix} \cos(t\frac{\pi}{2} & \sin(t\frac{\pi}{2}) \\ -\sin(t\frac{\pi}{2} & \cos(t\frac{\pi}{2}) \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} (1-q) + \cos(t\frac{\pi}{2})q & \sin(t\frac{\pi}{2})v \\ -\sin(t\frac{\pi}{2})v^* & (1-p) + \cos(t\frac{\pi}{2})p \end{pmatrix}.$$

Schließlich ist

$$\begin{aligned} u \begin{pmatrix} p & \\ & 0 \end{pmatrix} u^* &= \begin{pmatrix} v & -(1-q) \\ 1-p & v^* \end{pmatrix} \cdot \begin{pmatrix} p & \\ & 0 \end{pmatrix} \cdot \begin{pmatrix} v^* & 1-p \\ -(1-q) & v \end{pmatrix} \\ &= \begin{pmatrix} vpv^* & \\ & 0 \end{pmatrix} = \begin{pmatrix} q & \\ & 0 \end{pmatrix}, \end{aligned}$$

also $\left(\begin{smallmatrix} p & \\ & 0 \end{smallmatrix}\right) \sim_{U_0} \left(\begin{smallmatrix} q & \\ & 0 \end{smallmatrix}\right)$ in $M_2(A)$; (Proposition II.6) liefert jetzt, dass tatsächlich $\left(\begin{smallmatrix} p & \\ & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} q & \\ & 0 \end{smallmatrix}\right)$ in $M_2(A)$. □

Lemma II.8: *Seien A eine unitale C^*-Algebra, p, q, p', q' Projektionen in A, sodass $p \perp q$, $p' \perp q'$ (d. h. $pq = 0$, $p'q' = 0$), $p \sim p'$ und $q \sim q'$. Dann gilt $p + q \sim p' + q'$.*

Beweis: Seien $v, w \in A$ mit $v^*v = p$, $vv^* = p'$, $w^*w = q$, $ww^* = q'$. Dann ist

$$w^*v = w^*ww^*vv^*v = 0 = wv^*,$$

d. h. $(v+w)^*(v+w) = v^*v+w^*w = p+q$, $(v+w)(v+w)^* = vv^*+ww^* = p'+q'$. □

Kapitel III.

Definition of K_0

Definition III.1: For a C^*-algebra A consider the embedding

$$M_n(A) \hookrightarrow M_{n+1}(A), \qquad x \longmapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

and put $M_\infty(A) := \bigcup_{n\in\mathbb{N}} M_n(A) = \{(a_{i,j})_{i,j\in\mathbb{N}} \mid a_{i,j} \in A, \text{only finitely many } a_{i,j} \neq 0\}$.

Bemerkung III.2: (i) If A is a C^*-algebra, then also $M_n(A) := \{(a_{i,j})_{1\leq i,j\leq n} \mid a_{i,j} \in A\}$. Indeed: Let $\pi\colon A \hookrightarrow B(H)$ be a faithful representation and define

$$\pi^{(n)}\colon M_n(A) \longrightarrow B(H^n) \cong M_n(B(H)), \qquad (a_{i,j}) \longmapsto (\pi(a_{i,j})).$$

Then $\pi^{(n)}$ is a faithful representation. Put $\|(a_{i,j})\| := \|\pi(a_{i,j})\|_{B(H^n)}$. With this norm, $M_n(A)$ becomes a C^*-algebra and we have

$$\max_{1\leq i,j\leq n} \|a_{i,j}\| \leq \|(a_{i,j})\| \leq \sum_{i,j=1}^{n} \|a_{i,j}\|.$$

(ii) We have $M_n(A) \subseteq M_\infty(A)$, i.e., we have a C^*-norm on $M_\infty(A)$ via $\|x\| = \|x\|_{M_n(A)}$ for $x \in M_n(A) \hookrightarrow M_\infty(A)$, but $M_\infty(A)$ is not complete with respect to this norm, i.e., $M_\infty(A)$ is not a C^*-algebra itself.

Definition III.3: For a C^*-algebra A put

$$H(A) := \{[p]_\sim \mid p \in M_\infty(A) \text{ is a projection}\}.$$

Lemma III.4: (i) $[\cdot]$ *is the equivalence class with respect to* $\sim$, $\sim_h$, $\sim_U$, $\sim_{U_0}$.

(ii) $H(A)$ *is an abelian semigroup via* $[p]+[q] := [p'+q']$, *where* $p \sim p'$, $q \sim q'$ *with* $p' \perp q'$ *(i.e.,* $p'q' = 0$*). The neutral element is* $[0]$.

Beweis: (i) Let $p, q \in M_\infty(A)$, i.e., $p, q \in M_n(A)$ for some $n \in \mathbb{N}$. If $p \sim q$ in $M_n(A)$ we have $p \sim_h q$ in $M_{2n}(A) \subseteq M_\infty(A)$ via Proposition II.7.

If now $p \sim_h q$ in $M_n(A)$, then $p \sim_h q$ in $M_2(M_n(A)) \cong M_{2n}(A) \subseteq M_\infty(A)$ and thus via Proposition II.6 we have $p \sim q$ in $M_{2n}(A) \subseteq M_\infty(A)$.

(ii) There are p', q' such that $p = p' := \left(\begin{smallmatrix} p & \\ & 0 \end{smallmatrix}\right)$, $q' := \left(\begin{smallmatrix} 0 & \\ & q \end{smallmatrix}\right) \sim q$ and independent of the choice of p' and q' let p, p', p'', q, q', q'' be projections in $M_\infty(A)$ with $p \sim p' \sim p''$, $q \sim q' \sim q''$ and $p' \perp q'$, $p'' \perp q''$.

Now Lemma II.8 gives $p' + q' \sim p'' + q''$, thus $[p' + q'] = [p'' + q'']$.

Associativity and commutativity are easily checked, just like the fact that $[0]$ is the neutral element. □

Definition III.5: Let H be an abelian semigroup and let $\Delta := \{(x, x) \mid x \in H\}$. Then $G(H) := H \times H/\Delta$ is called the *Grothendieck group of H*. Here

$$(a_1, b_1) \sim (a_2, b_2) :\Longleftrightarrow \exists\, z_1, z_2 \in H : (a_1 + z_1, b_1 + z_1) = (a_2 + z_2, b_2 + z_2).$$

In the following, we write $(a, b)^\bullet$ for elements in $G(H)$.

Lemma III.6: *Let H be an abelian semigroup and $G(H)$ be its Grothendieck group. Then $G(H)$ is indeed an abelian group with neutral element $(x, x)^\bullet$ and inverse element (b, a), given $(a, b)^\bullet$.*

Beweis: We have $(a, b)^\bullet + (x, x)^\bullet = (a + x, b + x)^\bullet = (a, b)^\bullet$ and $(a, b)^\bullet + (b, a)^\bullet = (a + b, a + b)^\bullet$ is the neutral element. Furthermore, since H already was abelian, we have

$$(a, b)^\bullet + (c, d)^\bullet = (a + c, b + d)^\bullet = (c + a, d + b)^\bullet = (c, d)^\bullet + (a, b)^\bullet,$$

thus $G(H)$ is an abelian group. □

Beispiel III.7: If we take the abelian semigroup $(\mathbb{N}, +)$, the Grothendieck group $G(\mathbb{N})$ is nothing else but $(\mathbb{Z}, +)$.

Lemma III.8: (i) *The Grothendieck map*

$$\varphi\colon H \longrightarrow G(H), \qquad a \longmapsto (a + x, x)^\bullet$$

is a semigroup homomorphism. It is neither injective nor surjective.

(ii) *The Grothendieck group has the following universal property: If* $\psi\colon H \to G$ *is a semigroup homomorphism, where* G *is a group, there is one and only one group homomorphism*

$$\alpha\colon G(H) \longrightarrow G$$

with $\alpha((a,b)^\bullet) = \psi(a) - \psi(b)$ *and* $\alpha \circ \varphi = \psi$*. In particular,* $G(H) = \{\varphi(a) - \varphi(b) \mid a, b \in H\}$*.*

(iii) *We have functoriality, i.e., for all semigroup homomorphisms* $\alpha\colon H \to H'$ *there is one and only one* $G(\alpha)\colon G(H) \to G(H')$ *such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\alpha} & H' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ G(H) & \xrightarrow[G(\alpha)]{} & G(H') \end{array}$$

is commutative.

Beweis: (i) We have

$$\varphi(a+b) = (a+b+x, x)^\bullet = (a+b+x+x, x+x)^\bullet = (a+x, x)^\bullet + (b+x, x)^\bullet = \varphi(a) + \varphi(b).$$

(ii) The map α is indeed a group homomorphism, since

$$\begin{aligned} \alpha((a,b)^\bullet + (c,d)^\bullet) &= \alpha((a+c, b+d)^\bullet) \\ &= \psi(a+c) - \psi(b+d) \\ &= \psi(a) - \psi(b) + \psi(c) - \psi(d) = \alpha((a,b)^\bullet) + \alpha((c,d)^\bullet). \end{aligned}$$

Furthermore it holds

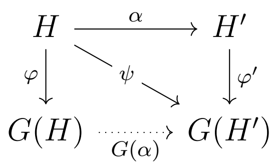
$$(\alpha \circ \varphi)(a) = \alpha((a+x, x)^\bullet) = \psi(a+x) - \psi(x) = \psi(a).$$

Suppose there was α' such that $\alpha' \circ \varphi = \psi$, then

$$\begin{aligned} \alpha((a,b)^\bullet) = \psi(a) - \psi(b) &= \alpha'(\varphi(a)) - \alpha'(\varphi(b)) \\ &= \alpha'((a+x, x)^\bullet) - \alpha'((b+x, x)^\bullet) \\ &= \alpha'((a+x, x)^\bullet) + \alpha'((x, b+x)^\bullet) \\ &= \alpha'((a+x+x, b+x+x)^\bullet) = \alpha'((a,b)^\bullet) \end{aligned}$$

and $\varphi(a) - \varphi(b) = (a,b)^\bullet$.

(iii) Using (ii), we define $\psi := \varphi' \circ \alpha$ and thus get a map $G(\alpha)$ such that it holds $G(\alpha) \circ \varphi = \psi = \varphi' \circ \alpha$, i.e., we are in the situation



□

Definition III.9: Es sei $H(A) := \{[p] \mid p \in \bigcup_{n\in\mathbb{N}} M_n(A) \text{ Projektion}\}$, $H(A)$ ist eine Halbgruppe und die Äquivalenzklassen sind bezüglich $\sim, \sim_U, \sim_{U_0}, \sim_h$. Dann ist

$$K_0(A) = G(H(A)) = H(A) \times H(A)/\text{diagonal}.$$

Berechnung III.10: (i) $K_0(\{0\}) = \{0\}$.

(ii) $K_0(\mathbb{C}) = \mathbb{Z}$.

We have $H(\mathbb{C}) = \{[p] \mid p \in M_\infty(\mathbb{C}) \text{ is a projection}\} = \mathbb{N}_0$, since projections $p, q \in M_n(\mathbb{C})$ are Murray-von Neumann equivalent if and only if $\operatorname{rank} p = \operatorname{rank} q$.

Kapitel IV.

Functoriality, additivity, finite stability, homotopy invariance and the standard picture of K_0

Eigenschaft IV.1: The map

$$K_0\colon \{\text{unital } C^*\text{-algebras}\} \longrightarrow \{\text{abelian groups}\}$$

is functorial, i.e.,

(i) If $\varphi\colon A \to B$ is a unital *-homomorphism, then there is a group homomorphism $K_0(\varphi) = \varphi_*\colon K_0(A) \to K_0(B)$,

(ii) If we have unital *-homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, then it holds $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

Beweis: (i) Given a unital *-homomorphism $\varphi\colon A \to B$, $a \mapsto \varphi(a)$, we get homomorphisms

$$\begin{aligned} M_\infty(\varphi)\colon M_\infty(A) &\longrightarrow M_\infty(B), & (a_{i,j}) &\longmapsto (\varphi(a_{i,j})) \\ H(\varphi)\colon H(A) &\longrightarrow H(B), & [p] &\longmapsto [M_\infty(\varphi)(p)] \\ K_0(\varphi)\colon K_0(A) &\longrightarrow K_0(B), & ([p],[q])^\bullet) &\longmapsto ([M_\infty(\varphi)(p)],[M_\infty(\varphi)(q)]) \end{aligned}$$

Eigenschaft IV.2: The functor K_0 is additive, i.e., $K_0(A\oplus B) = K_0(A)\oplus K_0(B)$.

Beweis: The direct sum $A \oplus B = \{(a,b) \mid a \in A, b \in B\}$ is a unital C^*-algebra via the pointwise operations and the norm $\|(a,b)\| := \max\{\|a\|,\|b\|\}$.

It holds

$$\{((a_{i,j},b_{i,j}))_{1\le i,j\le n}\} = M_n(A\oplus B) \cong M_n(A)\oplus M_n(B) = \{((a_{i,j})_{1\le i,j\le n},(b_{i,j})_{1\le i,j\le n}\},$$

thus $M_\infty(A \oplus B) \cong M_\infty(A) \oplus M_\infty(B)$, $H(A \oplus B) \cong H(A) \oplus H(B)$ and thus via Lemma III.8 (ii), we get $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$. □

Eigenschaft IV.3: The functor K_0 is finitely stable, i.e., $K_0(M_n(A)) = K_0(A)$.

Beweis: We have injections

$$j\colon A \hookrightarrow M_n(A), \qquad x \longmapsto \begin{pmatrix} x & \\ & 0 \end{pmatrix},$$

thus $M_\infty(j)\colon M_\infty(A) \to M_\infty(M_n(A)) \cong M_\infty(A)$ is a bijection. Hence $H(M_n(A)) \cong H(A)$ and $K_0(A) \cong K_0(M_n(A))$. □

Berechnung IV.4: (i) $K_0(M_n(\mathbb{C})) = K_0(\mathbb{C}) = \mathbb{Z}$,

(ii) $K_0(\mathbb{C}^n) = K_0(\bigoplus_{i=1}^n \mathbb{C}) = \bigoplus_{i=1}^n K_0(\mathbb{C}) = \mathbb{Z}^n$.

(iii) If A is a finite-dimensional C^*-algebra[1], then $A \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ and thus $K_0(A) = \mathbb{Z}^k$.

Definition IV.5: Let A and B be unital C^*-algebras

(i) If α and $\beta\colon A \to B$ are unital *-homomorphisms and there are unital *-homomorphisms $(\varphi_t)_{t\in[0,1]}$, where $\varphi_t\colon A \to B$ for $t \in [0,1]$, such that $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$, $\varphi_0 = \alpha$ and $\varphi_1 = \beta$, we call α and β *homotopic* and write $\alpha \sim_h \beta$.

(ii) If there are unital *-homomorphisms $\alpha\colon A \to B$, $\beta\colon B \to A$ such that $\alpha \circ \beta \sim_h \mathrm{id}_B$, $\beta \circ \alpha \sim_h \mathrm{id}_A$, we call A and B *homotopic* and write $A \sim_h B$.

(iii) If $A \sim_h 0$, we call A *contractible.*

Bemerkung IV.6: Two unital *-homomorphisms α, β are homotopic if and only if there is a map $\varphi\colon A \to C([0,1], B) := \{f\colon [0,1] \to B \text{ continuous}\}$ with $\varphi \circ \mathrm{ev}_0 = \alpha$, $\varphi \circ \mathrm{ev}_1 = \beta$, where

$$\mathrm{ev}_x\colon C([0,1], B) \longrightarrow B, \qquad f \longmapsto f(x).$$

Lemma IV.7: *Let A be a unital C^*-algebra.*

(i) *We have $A \sim_h C([0,1], A)$,*

(ii) *We have $C_0((0,1], A) \sim_h 0$, $CA := C_0([0,1), A) \sim_h 0$.*

[1] Those are always unital.

Beweis: (i) The set $C([0,1], A) := \{f : [0,1] \to A \text{ continuous}\}$ is a C^*-algebra with pointwise operations and $\|f\|_\infty := \sup_{t\in[0,1]} \|f(t)\|_\infty$. Consider the maps

$$\begin{aligned} \alpha\colon A &\longrightarrow C([0,1], A), & a &\longmapsto (t \mapsto a), \\ \beta\colon C([0,1], A) &\longrightarrow A, & f &\longmapsto f(0), \\ \varphi_t\colon C([0,1], A) &\longrightarrow C([0,1], A), & \varphi_t(f)(s) &:= f(ts). \end{aligned}$$

Then $\beta\circ\alpha = \mathrm{id}_A$, $\varphi_0 = \alpha\circ\beta$ and $\varphi_1 = \mathrm{id}_{C([0,1],A)}$, i.e., it holds $\beta\circ\alpha = \mathrm{id}_A \sim_h \mathrm{id}_A$ and $\alpha \circ \beta \sim_j \mathrm{id}_{C([0,1]),A)}$.

(ii) The set $C_0((0,1], A) = \{f : (0,1] \to A \text{ continuous} \mid f(0) = 0\}$ is a non-unital C^*-algebra with pointwise operations. Similarly as above, consider the maps

$$\begin{aligned} \alpha\colon 0 &\longrightarrow C([0,1], A), & 0 &\longmapsto (t \mapsto 0), \\ \beta\colon C([0,1], A) &\longrightarrow 0, & f &\longmapsto f(0), \\ \varphi_t\colon C([0,1], A) &\longrightarrow C([0,1], A), & \varphi_t(f)(s) &:= f(ts). \end{aligned}$$

Then $\beta \circ \alpha = \mathrm{id}_0$ and $0 = \alpha \circ \beta \sim_h \mathrm{id}_{C_0((0,1],A)}$. □

Eigenschaft IV.8: The functor K_0 is homotopy invariant, i.e.,

(i) If $\alpha \sim_h \beta$, then $K_0(\alpha) \sim_h K_0(\beta)$,

(ii) If $A \sim_h B$, then $K_0(A) \sim_h K_0(B)$.

Beweis: (i) Take $(\varphi_t)_{t\in[0,1]}$, where $\varphi_0 = \alpha$ and $\varphi_1 = \beta$. Then $[M_\infty(\varphi_t)(p)] = [M_\infty(\varphi_0)(p)]$, since $[\cdot]$ is with respect to „$\sim_h$“, i.e., $H(\varphi_t)$ is contant and thus $K_0(\varphi_t)$ is constant.

(ii) Let $\alpha\colon A \to B$, $\beta\colon B \to A$ with $\alpha\circ\beta \sim_h \mathrm{id}$, $\beta\circ\alpha \sim_h \mathrm{id}$. By (i), we thus have

$$K_0(\alpha) \circ K_0(\beta) = \mathrm{id}_{K_0(B)}, \qquad K_0(\beta) \circ K_0(\alpha) = \mathrm{id}_{K_0(A)}\,.$$ □

Berechnung IV.9: (i) It holds $K_0(C([0,1], A)) = K_0(A)$ via (4.7), (4,8); in particular it holds $K_0(C([0,1])) = \mathbb{Z}$.

(ii) If A is contractible, then $K_0(A) = 0$. In particular it holds $K_0(C_0((0,1], A)) = K_0(CA) = K_0(C_0((0,1]) = 0$.

Definition IV.10: Let A be a not necessarily unital C^*-algebra. Then we define $K_0'(A) := \ker K_0'(\sigma) \subseteq K_0(\tilde{A})$, where

$$\sigma\colon \tilde{A} \longrightarrow \mathbb{C}, \qquad (a,\lambda) \longmapsto \lambda$$

and $\tilde{A}$ is the unitalisation of A. Here $K_0'(\sigma)\colon K_0'(\tilde{A}) \to \mathbb{Z}$

Lemma IV.11: *Let A be a unital C^*-algebra. Then the definitions (4.10) and (3.9) coincide.*

Beweis: If A is unital, then $\tilde{A} \cong A \oplus \mathbb{C}$ as C^*-algebras. Then $K_0'(\tilde{A}) = K_0(A) \oplus K_0'(\mathbb{C})$ and $\ker K_0'(\sigma) = K_0(A)$, since

$$\sigma\colon A \oplus \mathbb{C} \longrightarrow \mathbb{C}, \qquad (a,\lambda) \longmapsto \lambda$$

and $\ker \sigma = A \triangleleft \tilde{A}$. □

Since K_0' and K_0 coincide, we omit the „$'$“ in the following.

Bemerkung IV.12: K_0 (from Definition (4.10)) has all properties (4.1), (4.2), (4.3), (4.8) also for non-unital C^*-algebras.

Proposition IV.13 (Standard picture):

(i) $K_0(A) = \{([p],[1_n])^\bullet \mid [p],[1_n] \in H(\tilde{A}), \sigma(p) \sim 1_n \in M_n(\mathbb{C})\}$, $\sigma\colon \tilde{A} \to \mathbb{C}$,

(ii) *It holds $([p],[1_n])^\bullet = 0$ if and only if there is some natural number k such that $\left(\begin{smallmatrix} p & \\ & 1_k \end{smallmatrix}\right) \sim 1_{n+k}$,*

(iii) *It holds $(([p],[1_n])^\bullet = ([q],[1_m])^\bullet$ if and only if there is some natural number k such that $\left(\begin{smallmatrix} p & \\ & 1_{m+k} \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} q & \\ & 1_{n+k} \end{smallmatrix}\right)$.*

Beweis: (i) „$\supseteq$“: Let $[p]$, $[1_n] \in H(\tilde{A})$, $\sigma(p) \sim 1_n$. Then

$$K_0(\sigma)([p],[1_n])^\bullet = ([\sigma(p)],[\sigma(1_n)])^\bullet = ([1_n],[1_n])^\bullet = 0,$$

i.e., $([p],[1_n])^\bullet \in \ker K_0(\sigma) = K_0(A)$.

„$\subseteq$“: Let $([p],[q])^\bullet \in K_0(A) = \ker K_0(\sigma)$ and let $p,q \in M_n\tilde{A}$. Put $v := \left(\begin{smallmatrix} q & 1-q \\ 1-q & q \end{smallmatrix}\right) \in M_{2n}\tilde{A}$. Then v is unitary and it holds $v\left(\begin{smallmatrix} q & \\ & 1-q \end{smallmatrix}\right)v^* = \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right)$ and thus $\left(\begin{smallmatrix} q & \\ & 1-q \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1_n & \\ & 0 \end{smallmatrix}\right)$. Now this implies

$$([p],[q])^\bullet = \left(\left[\begin{pmatrix} p & \\ & 1-q \end{pmatrix}\right], \left[\begin{pmatrix} q & \\ & 1-q \end{pmatrix}\right]\right)^\bullet = \left(\left[\begin{pmatrix} p & \\ & 1-q \end{pmatrix}\right], \left[\begin{pmatrix} 1_n & \\ & 0 \end{pmatrix}\right]\right)^\bullet$$

and $0 = ([\sigma\left(\begin{smallmatrix} p & \\ & 1-q \end{smallmatrix}\right)],[\sigma(1_n)])^\bullet$, i.e., $\sigma\left(\begin{smallmatrix} p & \\ & 1-q \end{smallmatrix}\right) \sim \sigma(1_n) = 1_n$.

(ii) „$\Leftarrow$“: We have

$$([p],[1_n])^\bullet = \left(\left[\begin{pmatrix} p & \\ & 1_k \end{pmatrix}\right], [1_{n+k}]\right)^\bullet = ([1_{n+k}],[1_{n+k}])^\bullet = 0.$$

„$\Rightarrow$“: If $([p],[1_n])^\bullet = 0$, then there is some projection $q \in M_k(\tilde{A})$ such that $[\left(\begin{smallmatrix} p & \\ & q \end{smallmatrix}\right)] = [\left(\begin{smallmatrix} 1_n & \\ & q \end{smallmatrix}\right)]$, i.e.,

$$\begin{pmatrix} p & \\ & 1_k \end{pmatrix} \sim \begin{pmatrix} p & & \\ & q & \\ & & 1_k - q \end{pmatrix} \sim \begin{pmatrix} 1_n & & \\ & q & \\ & & 1_k - q \end{pmatrix} \sim 1_{n+k}.$$

(iii) „$\Leftarrow$“: It holds

$$([p],[1_n])^\bullet = \left(\left[\begin{pmatrix} p & \\ & 1_{m+k} \end{pmatrix}\right], [1_{n+m+k}]\right)^\bullet = \left(\left[\begin{pmatrix} q & \\ & 1_{n+k} \end{pmatrix}\right], [1_{n+m+k}]\right)^\bullet = ([q],[1_m])^\bullet.$$

„$\Rightarrow$“: Suppose there are projections $r, s \in M_n\tilde{A}$ such that $([p]+[r],[1_n]+[r]) = ([q]+[s],[1_m]+[s])$. Then $\left(\begin{smallmatrix} p & \\ & r \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} q & \\ & s \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1_n & \\ & r \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1_m & \\ & s \end{smallmatrix}\right)$ and thus

$$\begin{pmatrix} p & & \\ & 1_m & \\ & & 1_k \end{pmatrix} \sim \begin{pmatrix} p & & & \\ & 1_m & & \\ & & r & \\ & & & 1_{k-r} \end{pmatrix} \sim \begin{pmatrix} q & & & \\ & 1_m & & \\ & & s & \\ & & & 1_{k-r} \end{pmatrix}.$$

which concludes the proof. □

Kapitel V.

Continuity and stability of K_0

Proposition V.1: *Let*

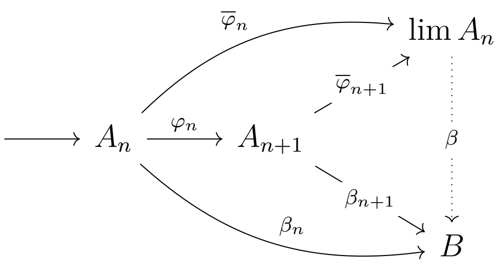
$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \qquad \dots \qquad A_n \xrightarrow{\varphi_n} A_{n+1} \longrightarrow \dots$$

be an „inductive system“ of objects $(A_n)_{n\in\mathbb{N}}$ and morphisms $(\varphi_n)_{n\in\mathbb{N}}$, here

(i) *A_n be C^*-algebras and φ_n be *-homomorphisms,*

(ii) *A_n be groups and φ_n be group homomorphisms.*

There exists a unique element (up to unique isomorphism) $\varinjlim_{\varphi_n} A_n$, the „inductive limit“ and morphisms $\overline{\varphi}_n\colon A_n \to \lim A_n$ in the cases (i) and (ii) with the following universal property: For all (β_n) and B there is one and only one β that renders commutative the following diagram:



In general, the inductive limit does not exist in arbitrary categories.

Beweis: We define $\mathfrak{A} := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in A_n, \exists\, N \in \mathbb{N}\, \forall\, n \geq N : x_{n+1} = \varphi_n x_n\}$, an equivalence relation on sequences via „$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if there is a natural number N such that for all $n \geq N$ it holds $x_n = y_n$“, a norm on $\mathfrak{A}/\sim$ via $\|[(x_n)_{n\in\mathbb{N}}]\| := \lim\|x_n\|_{A_n}$ for $[(x_n)_{n\in\mathbb{N}}] \in \mathfrak{A}/\sim$, $\lim A_n := \mathfrak{A}/\sim/\{[(x_n)_{n\in\mathbb{N}}] \mid \|[(x_n)_{n\in\mathbb{N}}]\| = 0\}$ and $\overline{\varphi}_n(x) := [(0,\dots,0,x,\varphi_n(x),\varphi_{n+1}(\varphi_n(x)),\dots)]$ for $x \in A_n$.

Then we have the following properties: „$\sim$“ is an equivalence relation, for a sequence $(x_n)_{n\in\mathbb{N}} \in \mathfrak{A}$ and $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$ it holds that for all $n \geq N$ we have $\|x_{n+1}\|_{A_{n+1}} = \|\varphi_n(x_n)\|_{A_{n+1}} \leq \|x_n\|_{A_n}$, i.e., $\lim\|x_n\|$ exists.

$\|\cdot\|$ is a C^*-seminorm on $\mathfrak{A}/\sim$, $\lim A_n$ is a C^*-algebra, $\overline{\varphi}_n(x) \in \lim A_n$ and $\overline{\varphi}_n \colon A_n \to \lim A_n$ is a *-homomorphism.

Now to the construction of β: Let $[(x_n)_{n\in\mathbb{N}}] \in \lim A_n$, i.e., there is a natural number N such that for all $n \geq N$ it holds $x_{n+1} = \varphi_n(x_n)$. Thus for $n \geq N$ it also holds $\beta_n(x_n) = \beta_{n+1}(\varphi_n(x_n)) = \beta_{n+1}(x_{n+1})$. This allows us to define

$$\beta([(x_n)_{n\in\mathbb{N}}]) := \lim_{n\to\infty} \beta_n(x_n) = \beta_N(x_N),$$

and now it holds $\beta\overline{\varphi}_n = \beta_n$, the uniqueness can be seen from the diagrams.

For the uniqueness of $\lim A_n$: If we had two limits $\lim A_n$, A', the universal properties of each of them granted that the homomorphisms β and β' were inverse isomorphisms, i.e., $\beta \circ \beta' = \mathrm{id}_{\lim A_n}$.

For groups, we don't need norms and can do a similar construction. □

Korollar V.2: *If we have a chain of C^*-algebras $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$, then $\lim A_n = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} A_n) \subseteq A$ is the direct limit where $\varphi_n \colon A_n \to A_{n+1}$, $x \mapsto x$.*

Beweis: We have to show that $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} A_n)$ has the universal property from Proposition V.1. Define $\beta\colon \mathrm{cl}(\bigcup_{n\in\mathbb{N}} A_n) \to B$ via $\beta|_{A_n} = \beta_n$ and extend. □

Proposition V.3: *Let (A_n, φ_n) be an inductive system.*

(i) *For $x \in \lim A_n$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \overline{\varphi}_{N_n}(A_{N_n})$ and $x_n \to x$, i.e., $\lim A_n = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} \overline{\varphi}_n(A_n))$.*

(ii) *If $\bigcup_{n\in\mathbb{N}} \beta_n(A_n) \subseteq B$ is dense, then β is surjective. If all β_n are injective, so is β.*

(iii) *If all A_n are simple, so is $\lim A_n$*

(iv) *Also $\varphi_n^{(m)} \colon M_m(A_n) \to M_{m+1}(A_{n+1})$, $(a_{i,j}) \mapsto (\varphi_n(a_{i,j}))$ is an inductive system and $\lim M_m(A_n) = M_m(\lim A_n)$.*

(v) *It holds $\widetilde{\lim A_n} = \lim \widetilde{A}_n$.*

Beweis: (i) Let $x \in \lim A_n = \mathrm{cl}_{\|\cdot\|}(\mathfrak{A}/\sim/\{\|[(x_n)]\| = 0\})$. Then there are sequences $(y^k)_{k\in\mathbb{N}} \subseteq \mathfrak{A}/\sim$ with $y^k \to x$ and $y^k = [(y_n^k)_{n\in\mathbb{N}}]$. For y^k there is a natural number N_k such that for all $n \geq N_k$ it holds $y^k_{n+1} = \varphi_n y^k_n$. Put $x_k := \overline{\varphi}_{N_k}(y^k_{N_k}) \in \overline{\varphi}_{N_k}(A_{N_k})$. Then

$$x_k = [(0, \dots, 0, y^k_{N_k}, \varphi_{N_k} y^k_{N_k}, \varphi_{N_k+1}\varphi_{N_k} y^k_{N_k}, \dots],$$
$$y^k = [(y^k_1, \dots, y^k_{N_k-1}, y^k_{N_k}, \varphi_{N_k} y^k_{N_k}, \varphi_{N_k+1}\varphi_{N_k}, \dots],$$

i.e., $x_k = y^k \to x$.

(ii) Since $\bigcup_{n\in\mathbb{N}} \beta_n(A_n) \subseteq \beta(\lim A_n)$ and $\beta(\lim A_n)$ is closed, β is surjective.

If all β_n are injective, we have: If $0 = \beta([(x_n)]) = \beta_N(x_N)$, then $x_N = 0$ since β_N is injective and thus

$$[(x_n)] = [(x_1, \dots, x_N, \varphi_N x_N, \varphi_{N+1}\varphi_N x_N, \dots)] = [0].$$

Assertions (iii) - (v) are left as exercises for the reader. □

Definition V.4: Let (A_n, φ_n) be an inductive system where all A_n are finite-dimensional. Then $\lim A_n$ is called an *approximately finite-dimensional algebra*.

Beispiel V.5: (i) The family $(M_n(\mathbb{C}))_{n\in\mathbb{N}}$ with $M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}), x \mapsto x$ has inductive limit $\lim M_n(\mathbb{C}) = K(H)$, where H is a separable Hilbert space over $\mathbb{C}$.

(ii) The family $(M_{2n}(\mathbb{C}))_{n\in\mathbb{N}}$ with the inclusions

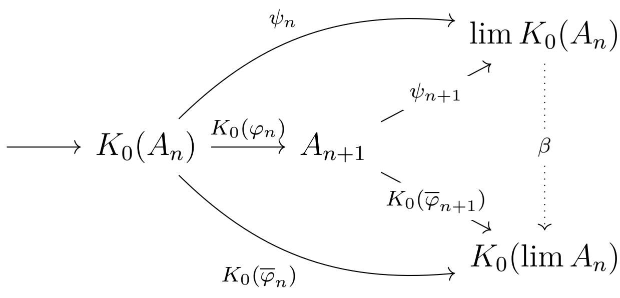
$$M_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{C}), \qquad x \longmapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

has inductive limit $\lim M_{2n}(\mathbb{C}) =: M_{2^\infty}(\mathbb{C})$, a so-called UHF algebra.

Eigenschaft V.6: The functor K_0 is continuous, i.e., it holds

$$K_0(\varinjlim_{\varphi_n} A_n) \cong \varinjlim_{K_0(\varphi_n)} K_0(A_n).$$

Beweis: We are in the situation



Making the simplifying assumptions that all A_n are unital and that all $\overline{\varphi}_n$ are injective, we have $\lim A_n = \operatorname{cl} \bigcup_{n\in\mathbb{N}} A_n$.

Let $([p],[1_n])^\bullet \in K_0(\lim A_n)$ where

$$p \in M_k(\lim A_n) = \lim M_k(A_n) = \operatorname{cl}(\bigcup_{n\in\mathbb{N}} M_k A_n)$$

is a projection. Then there is $(x_n)_{n\in\mathbb{N}}$, where $x_n \in M_k A_n$, such that $x_n \to p$. Put $h_n := x_n^* x_n$; then $h_n \to p^*p = p$ and $h_n^2 - h_n \to p^2 - p = 0$, i.e., there is some natural number N and some $0 \le h \in M_k(A_N)$ with $\|p-h\| < 1/2$ and $\|h^2 - h\| < 1/4$. Consider the function

$$f\colon [0,1/2) \cup (1/2,3/2] \longrightarrow \mathbb{R}, \qquad f(t) := \begin{cases} 0, & \text{if } t \in [0,1/2), \\ 1, & \text{if } t \in (1/2,3/2]. \end{cases}$$

Then $\|\mathrm{id} - f\|_\infty \le 1/2$. We have $1/2 \notin \operatorname{Sp} h \subseteq [0,3/2]$ since $\|h\| \le \|h-p\| + \|p\| \le 1/2 + 1$ and $h \ge 0$ and $\|\mathrm{id}_{\operatorname{Sp} h}^2 - \mathrm{id}_{\operatorname{Sp} h}\|_\infty = \|h^2 - h\| < 1/4$.

Hence, f is continuous on $\operatorname{Sp} h$ and $q := f(h) \in M_k(A_N)$ is a projection with $\|p - q\| \le \|p-h\| + \|h - f(h)\| < 1/2 + 1/2 = 1$, thus by (II.5), $p \sim_{U_0} q$ and thus by (II.6), $p \sim q$.

Thus $K_0(\overline{\varphi}_N)([q],[1_n])^\bullet = ([q],[1_n])^\bullet = ([p],[1_n])^\bullet$ and thus β is surjective.

Assume $\beta([p],[1_n])^\bullet = \beta([q],[1_m])^\bullet \in K_0(\lim A_n)$. Then, by (IV.13), we have

$$\begin{pmatrix} p & \\ & 1_{m+k} \end{pmatrix} \sim \begin{pmatrix} q & \\ & 1_{n+k} \end{pmatrix}$$

Thus, by II.6 and II.7 there is some unitary $u \in M_r \lim A_n$ with $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \sim_U \left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right)$. By functional calculus as above there is some unitary $u' \in M_r A_N$ such that $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \sim_U \left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right)$, hence by IV.13 we conclude $([p],[1_n])^\bullet = ([q],[1_m])^\bullet$. □

Eigenschaft V.7: The functor K_0 is stable, i.e., $K_0(A \otimes K) \cong K_0(A)$, where $K = K(H)$ for some separable Hilbert space.

Beweis: We can consider the sequences

$$\dots \longrightarrow M_n(\mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C}) \longrightarrow \dots \longrightarrow K$$

$$x \longmapsto \begin{pmatrix} x & \\ & 0 \end{pmatrix}$$

$$\dots \longrightarrow M_n(A) \longrightarrow M_{n+1}(A) \longrightarrow \dots \longrightarrow A \otimes K$$

$$x \longmapsto \begin{pmatrix} x & \\ & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} \dots \longrightarrow & K_0(M_n(A)) & \longrightarrow & K_0(M_{n+1}(A)) & \longrightarrow & \dots & \longrightarrow & K_0(A) \\ & \| & & \| & & & & \\ \dots \longrightarrow & K_0(A) & \xrightarrow[\text{id}]{} & K_0(A) & \longrightarrow & \dots & \longrightarrow & K_0(A) \end{array}$$

□

Berechnung V.8: (i) $K_0(K) = K_0(\mathbb{C}) = \mathbb{Z}$,

(ii) For the family $(M_{2n}(\mathbb{C}))_{n\in\mathbb{N}}$ from Beispiel V.5 (ii) we have the diagram

$$\begin{array}{ccccccc} K_0(M_2(\mathbb{C})) & \longrightarrow & K_0(M_4(\mathbb{C})) & \longrightarrow & K_0(M_8(\mathbb{C})) & \longrightarrow & \dots \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \mathbb{Z} & \xrightarrow[\cdot 2]{} & \mathbb{Z} & \xrightarrow[\cdot 2]{} & \mathbb{Z} & \xrightarrow[\cdot 2]{} & \dots \end{array}$$

and thus $K_0(M_{2^\infty}(\mathbb{C})) \cong \mathbb{Z}[1/2]$, where

$$Z[1/2] := \{m/2^k \mid m \in \mathbb{Z}, k \in \mathbb{N}\} \subseteq \mathbb{Q}.$$

(iii) For a separable Hilbert space H, we have $K_0(B(H)) = \{0\}$.

To see this, let $p \in M_n(B(H)) = B(H^n)$ be a projection. Then $p = \begin{pmatrix} p & \\ & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ in $M_{n+1}(B(H))$, but $\begin{pmatrix} p & \\ & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ since both are infinite projections. Hence $[p] + [\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}] = [\begin{pmatrix} p & \\ & 1 \end{pmatrix}] = [\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}]$, hence

$$([p],[q])^\bullet = ([p] + [\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}], [q] + [\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}])^\bullet = ([\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}], [\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}])^\bullet = 0.$$

(iv) If M is a factor of type I_∞, II_∞ or III, then $K_0(M) = 0$ as in (iii). Adding $\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ yields equivalent infinite projections.

(v) If M is a factor of type I_2, then $K_0(M) = \mathbb{R}$. We have the dimension function $D\colon \{\text{projections in } M\} \to \mathbb{R}$ and thus a function

$$D^*\colon H(M) \longrightarrow \mathbb{R}, \qquad [p] \longmapsto D_n(p),$$

where $p \in M_n(M)$ is a projection, which is well-defined since „If $p \sim q$, then $D_n(p) = D_n(q)$“ holds by construction and it is a semigroup homomorphism, since

$$D^*([p] + [q]) = D^*([p' + q']) = D^*([p']) + D^*([q']) = D^*([p]) + D^*([q]).$$

By the universal property there is one and only one group homomorphism $\alpha\colon K_0(M) \to \mathbb{R}$ such that

$$\alpha([p], [q])^\bullet = D^*([p]) - D^*([q]).$$

This map α is surjective, since for $r \in \mathbb{R}$ there are some natural number n and $t \in [0, D(1)]$ such that $nt = |r|$. For said t, we find a projection p_t with $D(p_t) = t$ and thus $D^*([\operatorname{diag}(p_t, \dots, p_t)]) = n$ and $D(p_t) = |r|$. Now

$$\alpha([(\operatorname{diag}(p_t, \dots, p_t)], 0)^\bullet = |r|, \qquad \alpha(0, [\operatorname{diag}(p_t, \dots, p_t)])^\bullet = -|r|.$$

This map α is injective, since we can calculate

$$D_n(p) - D_n(q) = \alpha([p], [q])^\bullet = \alpha([r], [s])^\bullet = D_n(r) - D_n(s),$$

i.e., $D_n(p' + s') = D_n(r' + q')$ and thus $p' + s' \sim r' + q'$. Hence we have

$$([p], [q])^\bullet = ([p] + [s], [q] + [s])^\bullet = ([p' + s'], [q] + [s])^\bullet = ([q] + [r], [q] + [s])^\bullet = ([r], [s])^\bullet.$$

Satz V.9: *Let A and B be approximately finite-dimensional algebras. Then $A \cong B$ holds if and only if*

$$(K_0(A), K_0^+(A), \Gamma(A)) \cong (K_0(B), K_0^+(B), \Gamma(B)).$$

Here, $K_0^+(A) = H(A)$ ordered per $[p] \leq [q]$ if and only if $p \sim q' \leq q$ and $\Gamma(A) = \{[p] \mid p \text{ is a projection in } A\}$.

Beispiel V.10: We have $K_0(M_n(\mathbb{C})) = \mathbb{Z}$, $K_0^+(M_n(\mathbb{C}) = \mathbb{N}$, $\Gamma(M_n(\mathbb{C})) = \{1, \dots, n\}$ and $M_2(\mathbb{C}) \ncong M_4(\mathbb{C})$ due to Γ.

Kapitel VI.

Half exactness and split exactness of K_0

Definition VI.1: Let $\ldots \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} \ldots$ be a system of objects and morphisms.

(i) The sequence is exact, if for all natural numbers n it holds $\ker \varphi_{n+1} = \operatorname{im} \varphi_n$,

(ii) The sequence $0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$ is exact if and only if ι is injective, π is surjective and $\ker \pi = \operatorname{im} \iota$; then called a *short exact sequence.*

(iii) The sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \underset{\pi}{\overset{\varphi}{\leftrightarrows}} B \longrightarrow 0$$

is *split exact*, if in addition $\pi \circ \varphi = \mathrm{id}_B$. The map φ is then called *split (map).*

Lemma VI.2: *The sequence* $0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$ *is a short exact sequence of C^*-algebras if and only if* $I \triangleleft A$ *and* $B \cong A/I$.

Beweis: If ι is injective, we have $I \subseteq A$ and $I = \operatorname{im} \iota = \ker \pi \triangleleft A$, thus, as π is surjective, $B \cong A \ker \pi = A/I$. □

Lemma VI.3: *If* $0 \longrightarrow I \xrightarrow{\iota} A \underset{\pi}{\overset{\varphi}{\leftrightarrows}} B \longrightarrow 0$ *is a split exact sequence of abelian groups, then* $A \cong I \oplus B$.

Beweis: The map $\alpha\colon I \oplus B \to A$, $(x, y) \mapsto \iota(x) + \varphi(y)$ is a group homomorphism. The homomorphism α is injective, since if $\alpha(x, y) = 0$, then $y = \pi(\iota(x) + \varphi(y) = 0$, i.e., $\iota(x) = 0$. Since ι is injective, this means $x = 0$ and thus $(x, y) = 0$.

Furthermore α is surjective: Let $z \in A$. If we put $x := z - \varphi(\pi(z)) \in \ker(\pi) = \mathrm{im}(\iota)$ and $y := \pi(z) \in B$, then $\alpha(\iota^{-1}(x), y) = z$. □

Bemerkung VI.4: The statement in Lemma VI.3 does not hold for C^*-algebras, i.e., $A \ncong I \oplus B$. For instance

$$0 \longrightarrow A \longrightarrow \tilde{A} \leftrightarrows \mathbb{C} \longrightarrow 0$$

but $\tilde{A} \ncong A \oplus \mathbb{C}$.

Lemma VI.5: *If the sequence* $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$ *is an exact sequence of groups, then it is also split exact*

Beweis: For $1 \in \mathbb{Z}$ we find $b \in A$ with $\pi(b) = 1$. Put $\varphi\colon \mathbb{Z} \to A$, $n \mapsto \sum_{i=1}^{n} b$ (or $n \mapsto \sum_{i=0}^{-n}(-b)$ if $n < 0$). □

Eigenschaft VI.6 (K_0 is half-exact): If $0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$ is a short exact sequence of C^*-algebras, then

$$K_0 I \xrightarrow{K_0(\iota)} K_0(A) \xrightarrow{K_0(\pi)} K_0(B)$$

is an exact sequence of groups, i.e., $\ker K_0(\pi) = \mathrm{im}\, K_0(\iota)$.

Beweis: „$\subseteq$“: Since $\pi \circ \iota = 0$ and K_0 is a functor, we have $\mathrm{im}\, K_0(\iota) \subseteq \ker K_0(\pi)$.

„$\supseteq$“: Let $([p],[1_n])^\bullet \in K_0(A)$ with $K_0(\pi)([p],[1_n])^\bullet = ([\pi(p)],[1_n])^\bullet = 0$. Via IV.13, there is some natural number k such that $\begin{pmatrix} \pi(p) & \\ & 1_k \end{pmatrix} \sim 1_{n+k}$. Put $p' := \begin{pmatrix} p & \\ & 1_k \end{pmatrix} \in M_N(\tilde{A})$, then $\pi(p') \sim 1_{n+k} \in M_N(\tilde{B})$. By II.6 and II.7, we find $u \in U_0(M_{2N}(\tilde{B}))$ with $u\pi(p')u^* = 1_{n+k}$ and since π is surjective, via I.4 we find $w \in U_0(M_{2N}(\tilde{A}))$ such that $\pi(w) = u$.

Now $\pi(wp'w^* - 1_{n+k}) = u\pi(p')u^* - 1_{n+k} = 0$, hence $wp'w^* - 1_{n+k} \in \ker\pi = \mathrm{im}\,\iota$. Thus there are $a, b \in \tilde{I}$ with $\iota(a) = wpw' - 1_{n+k}$, $\iota(b) = 1_{n+k}$. Put $q := a + b \in \tilde{I}$. Then $\iota(q) = wp'w^*$, where q is a projection, since $\iota(q)$ is a projection and ι is injective. We thus can calculate

$$([p],[1_n])^\bullet = ([p'],[1_{n+k}])^\bullet = ([\iota(q)],[1_{n+k}])^\bullet \in \mathrm{im}\, K_0(\iota). \qquad \square$$

Eigenschaft VI.7 (K_0 is split exact): If

$$0 \longrightarrow I \xrightarrow{\iota} A \underset{\pi}{\overset{\varphi}{\leftrightarrows}} B \longrightarrow 0$$

is a split exact sequence of C^*-algeras, then

$$0 \longrightarrow K_0 I \xrightarrow{K_0\iota} K_0 A \underset{K_0\pi}{\overset{K_0\varphi}{\leftrightarrows}} B \longrightarrow 0$$

is split exact, i.e., $K_0 A = K_0 I \oplus K_0 B$.

Beweis: Since $K_0(\pi) \circ K_0(\varphi) = K_0(\mathrm{id}) = \mathrm{id}_{K_0 B}$, $K_0(\varphi)$ is a split and $K_0(\pi)$ is surjective. Hence, by Eigenschaft VI.6, we need to show that $K_0(\iota)$ is injective.

Let therefore $([p], [1_n])^\bullet \in K_0(I)$ with $K_0([p], [1_n])^\bullet = 0$ and $p \in M_M \tilde{I}$, i.e., $p - 1_n \in M_M I$. By IV.13, there is some natural number k such that $\left(\begin{smallmatrix} \iota(p) & \\ & 1_k \end{smallmatrix}\right) \sim 1_{n+k}$. Put $p' := \left(\begin{smallmatrix} p & \\ & 1_k \end{smallmatrix}\right) \in M_M(\tilde{I})$; then $\iota(p') \sim 1_{n+k}$.

By II.6 and II.7, there is some $u \in U_0(M_{2M}\tilde{A})$ with $u\iota(p')u^* = 1_{n+k}$. Put $v := \varphi\pi u^* u \in M_{2M}(\tilde{A})$ with $\pi(v) = 1$. Then $v - 1 \in \ker \pi = \operatorname{im} \iota$, i.e., we have $w \in M_{2M}(\tilde{I})$ with $\iota(w) = v$ (in fact there is $w_0 \in M_{2M}\tilde{I}$ with $\iota(w_0) = v - 1$; put $w := w_0 + 1$).

Since v is a unitary and ι is injective, also w is unitary and

$$\begin{aligned}(\pi \circ \iota)(p') &= (\pi \circ \iota)\left(\begin{pmatrix} p & \\ & 1_k \end{pmatrix} - \begin{pmatrix} 1_n & \\ & 1_k \end{pmatrix}\right) \\ &= (\pi \circ \iota)(p - 1_n) + (\pi \circ \iota)(1_{n+k}) = 0 + 1_{n+k}.\end{aligned}$$

Thus $\iota(wp'w^*) = v\iota(p')v^* = \varphi\pi(u^*)u\iota(p')u^*\varphi\pi(u) = \varphi\pi(u^* 1_{n+k} u) = \varphi\pi\iota(p') = \varphi(1_{n+k}) = \iota(1_{n+k})$, which means —since ι is injective— that $wp'w^* = 1_{n+k}$, hence $\left(\begin{smallmatrix} p & \\ & 1_k \end{smallmatrix}\right) \sim 1_{n+k}$. By IV.13, we then get $([p], [1_n])^\bullet = 0$ in $K_0 I$. □

Berechnung VI.8: Let $A := \{f \in C([0,1], M_n(\mathbb{C}) \mid f(1) \in D_n \subseteq M_n(\mathbb{C})\} \subseteq C([0,1], M_n(\mathbb{C}))$, where D_n denotes the square matrices whose non-diagonal entries are zero. Then A is a C^*-algebra since it is a closed *-subalgebra. The sequence

$$0 \longrightarrow C_0([0,1), M_n(\mathbb{C})) \longrightarrow A \underset{\longleftarrow}{\longrightarrow} D_n \longrightarrow 0$$

is split-exact, thus

$$K_0(A) = K_0(C_0([0,1), M_n(\mathbb{C})) \oplus K_0(D_n) = K_0 \oplus K_0(\mathbb{C}^n) = \{0\} \oplus \mathbb{Z}^n.$$

Kapitel VII.

K_1

Definition VII.1: Let $U^\infty(\tilde{A}) := \bigcup_{n\in\mathbb{N}} U(M_n\tilde{A})$,

$$U(M_n(\tilde{A})) \hookrightarrow U(M_{n+1}\tilde{A}), \qquad x \longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix},$$

and $U_0^\infty(\tilde{A}) := \bigcup_{n\in\mathbb{N}} U_0(M_n(\tilde{A})$. Then $u \sim v$ if $v^*u \in U_0^\infty\tilde{A}$ defines an equivalence relation on $U^\infty(\tilde{A})$ and we call $K_1(A) := U^\infty(\tilde{A})/U_0^\infty(\tilde{A})$.

Lemma VII.2: *The following are equivalent:*

(i) $[u] = [v]$,

(ii) *There is some natural number n and there are $u, v \in M_n\tilde{A}$ with $u \sim_h v$ in $U_0(M_n\tilde{A})$ (i.e., $\gamma_t \in M_n\tilde{A}$).*

(iii) *$u \sim_h v$ in $U^\infty(\tilde{A})$ (i.e., $\gamma_t \in M_{n_t}\tilde{A}$).*

The set $K_1(A)$ turns into an abelian group via $[u][v] := [uv]$, $[u]^{-1} = [u^*]$, $e = [1]$.

Beweis: „(i) $\Rightarrow$ (ii)“: If $[u] = [v]$, then $v^*u \in U_0^\infty\tilde{A}$, i.e., $v^*u \in U_0(M_n\tilde{A})$ for some $n \in \mathbb{N}$. Hence, there is a path (γ_t) in $U(M_n\tilde{A})$ with $\gamma_0 = v^*u$, $\gamma_1 = 1$. Put $\tilde{\gamma}_t := v\gamma_t$. Then $\tilde{\gamma}_0 = u$, $\tilde{\gamma}_1 =)v$, hence $u \sim_h v$ in $M_n\tilde{A}$.

„(ii) $\Rightarrow$ (i)“: For (γ_t) in $U(M_n\tilde{A})$ with $\gamma_0 = u$ and $\gamma_1 = v$, put $\tilde{\gamma}_t = v^*\gamma_t$, thus $v^*u \sim_h 1$ and hence $[u] = [v]$.

„(ii) $\Rightarrow$ (iii)“ is clear.

For „(iii) $\Rightarrow$ (ii)“ let (γ_t) in $U(M_{n_t}\tilde{A})$ with $\gamma_0 = u$ and $\gamma_1 = v$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n =$ with $\|\gamma_{t_{i+1}} - \gamma_{t_i}\| < 2$. By I.4, we have $u = \gamma_{t_0} \sim_h \gamma_{t_1} \sim_h \cdots \sim_h \gamma_{t_n} = v$ via a new path in $U(M_{\max\{n_{t_0},\ldots,n_{t_n}\}}\tilde{A})$.

Finally, we want to show that $K_1(A)$ is an abelian group. We have by Lemma I.6 that

$$[u][v] = \left[\begin{pmatrix} uv & \\ & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} vu & \\ & 1 \end{pmatrix}\right] = [v][u].$$

□

Eigenschaft VII.3: K_1 is a functor and it is homotopy invariant.

Beweis: The homotopy invariance is clear by Lemma VII.2. If we have a morphism $\varphi\colon A \to B$, we get a homomorphism $\tilde{\varphi}\colon \tilde{A} \to \tilde{B}$ and thus homomorphisms $U^\infty \tilde{A} \to U^\infty \tilde{B}$ and $U_0^\infty \tilde{A} \to U_0^\infty \tilde{B}$, i.e., a morphism $K_1(A) \to K_1(B)$. □

Definition VII.4: Let A be a C^*-algebra. Then $SA := C_0((0,1), A) := A(0,1)$, where $C_0((0,1), A)$ is the set of continuous functions $f\colon (0,1) \to A$ with $f(0) = f(1) = 0$, is called *suspension of* A.

Lemma VII.5: *The suspension S is a functor, i.e., if $\varphi\colon A \to B$ is a *-homomorphism, then $S_\varphi\colon SA \to SB$ defined via*

$$S_\varphi(f)(t) := \varphi(f(t))$$

is a *-homomorphism with $S_{\varphi\circ\psi} = S_\varphi \circ S_\psi$.

Lemma VII.6:

(i) $\widetilde{SA} = \{f\colon [0,1] \to \tilde{A} \mid f = \lambda 1 + g, \lambda \in \mathbb{C}, g \in SA\}$, *i.e.,* $SA = \mathbb{C}1$.

(ii) $M_n(\widetilde{SA}) = \{f\colon [0,1] \to M_n\tilde{A} \mid f = x + g, x \in M_n(\mathbb{C}), g \in M_n(SA)\}$.

Satz VII.7: *It holds $K_0(SA) \cong K_1(A)$.*

Beweis: (1) Construction of $\alpha\colon K_1(A) \to K_0(SA)$: Let $u \in U(M_n\tilde{A})$. Hence, by Lemma I.6, $\begin{pmatrix} u & \\ & u^* \end{pmatrix} \sim_h 1$ via a path $(\gamma_t)_{t\in[0,1]}$ of unitaries. Put

$$p\colon [0,1] \longrightarrow M_{2n}(\tilde{A}), \qquad t \longmapsto \gamma_t \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \gamma_t^*;$$

this gives a path of projections with $p(0) = p(1) = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$, hence $p \in M_{2n}(\widetilde{SA})$. Now put $\alpha([u]) := ([p], [1_n])^\bullet \in K_0(SA)$.

(2) Construction of $\beta\colon K_0(SA) \to K_1(A)$: Let $([p], [1_n])^\bullet \in K_0(SA)$, hence $p \in M_m\widetilde{SA}$ is a projection and without loss of generality, $p(0) = p(1) = 1_n$. Thus, $p(0) \sim_h p(1)$ via $(p(t))_{t\in[0,1]}$. By II.6, there is some unitary $w \in M_m(\widetilde{SA})$ with $wp(0)w^* = p(1)$. Then $w\begin{pmatrix} 1_n & \\ & 0 \end{pmatrix} = wp(0) = p(1)w = \begin{pmatrix} 1_n & \\ & 0 \end{pmatrix} w$, i.e., $w = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$. Put $\beta([p], [1_n])^\bullet := [u] \in K_1(A)$.

Now one can show that indeed α and β are inverse to each other as maps which gives the assertion. □

Korollar VII.8: *The functor K_1 has the same properties as K_0.*

Berechnung VII.9: (i) $K_1(\mathbb{C}) = U^\infty(\mathbb{C})/U_0^\infty(\mathbb{C}) = \{0\}$ (in Lemma I.5, we have already shown that $U(M_n(\mathbb{C})) = U_0(M_n(\mathbb{C}))$.

(ii) Let M be a von Neumann-algebra. Then $K_1(M) = 0$ (if $u \in M$ is a unitary, then $u = \exp(ih)$ for some selfadjoint h by the measurable functional calculus and $\gamma_t := \exp(ith)$ does the trick).

(iii) $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$, since the sequence

$$0 \longrightarrow C(\mathbb{S}^1 - \{1\}) \longrightarrow C(\mathbb{S}^1) \underset{\text{constant fcts.}}{\rightleftarrows} \mathbb{C} \longrightarrow 0$$

$$f \longmapsto f(1)$$

is exact and thus, by Eigenschaft VI.7, we have

$$K_0(C(\mathbb{S}^1)) = K_0(S\mathbb{C}) \oplus K_0(\mathbb{C}) = K_1(\mathbb{C}) \oplus K_0(\mathbb{C}) = \mathbb{Z}.$$

(iv) As a small overview over our recent accomplishments and future goals:

	K_0	K_1
$\bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$	$\mathbb{Z}^k$	0
$K(H)$	$\mathbb{Z}$	0
Approximately finite-dimensional algebras	depends	0
$C([0,1])$	$\mathbb{Z}$	0
$A[0,1]$	$K_0(A)$	$K_1(A)$
$C_0((0,1])$	0	0
CA	0	0
$C_0((0,1))$	0	?
SA	$K_1(A)$	?
Type I_∞, II_∞, III factors	0	0
Type II_1 factors	$\mathbb{R}$	0
$\{f \in C([0,1], M_n(\mathbb{C})) \mid f(1) \text{ diag}\}$	$\mathbb{Z}^n$	0
$C(\mathbb{S}^1)$	$\mathbb{Z}$	?
$Q(H) = B(H)/K(H)$	?	?

Kapitel VIII.

Long exact sequences

Definition VIII.1: Let A be a C^*-algebra and X be a locally compact space. Then we denote $AX := C_0(X, A)$.

Lemma VIII.2: *The „X-functor“ is exact, i.e., if* $0 \to I \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ *is exact, then* $0 \to IX \to AX \to BX \to 0$ *is exact.*

Beweis: The map $IX \hookrightarrow AX$, $f \mapsto (t \mapsto \iota(f(t)))$ is injective.

Now for $\ker \pi_X = \operatorname{im} \iota_X$: Some function f is in the image of ι_X if and only if for all t it holds $f(t) \in \operatorname{im} \iota = \ker \pi$, i.e., if and only if $f \in \ker \pi_X$.

For the surjectivity of π_X: Let $f \in C_0(X)$, $x \in B$ and $y \in A$ with $\pi(y) = x$ (this is possible because π is surjective). Put $g(t) := f(t)x$ and $h(t) := f(t)y$. Hence $g \in BX$, $h \in AX$ and $\operatorname{im}(\pi_X) \ni \pi_X(h) = g$. By some partition of unity argument, we find that such functions g are linearly dense in BX which gives that $\pi_X(AX) \subseteq BX$ is dense, hence $\pi_X(AX) = BX$. □

Definition VIII.3: Let A, B be C^*-algebras and $\alpha\colon A \to B$ be a *-homomorphism. Then we put

$$C_\alpha := \{(x, y) \in A \oplus B([0, 1)) \mid \alpha(x) = y(0)\},$$
$$Z_\alpha := \{(x, y) \in A \oplus B([0, 1]) \mid \alpha(x) = y(0)\}.$$

The set C_α is called *mapping cone*, the set Z_α is called *mapping clyinder.*

Lemma VIII.4: *Let $\alpha\colon A \to B$ be a *-homomorphism.*

(i) *If α is surjective, then the sequences* $0 \to \ker \alpha \to A \to B \to 0$ *and*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\alpha & \longrightarrow & C_\alpha & \longrightarrow & B([0,1)) & \longrightarrow & 0 \\ & & x & \longmapsto & (x,0) & & & & \\ & & & & (x,y) & \longmapsto & y & & \end{array}$$

are exact.

(ii) *If* α *is injective, then* $C_\alpha \cong \{f \in B([0,1)) \mid f(0) \in \alpha(A)\}$.

(iii) *If* α *is surjective, then*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SB & \longrightarrow & C_\alpha & \longrightarrow & A & \longrightarrow & 0 \\ & & y & \longmapsto & (0,y) & & & & \\ & & & & (x,y) & \longmapsto & x & & \end{array}$$

is exact.