Refresher – math for MINT

§1 Functions, graphs, limits

[Funktion]

1.1 Def: (a) Let $D \subseteq \mathbb{R}$ be a set, let $f: D \to \mathbb{R}$ be a function,
i.e. for every $x \in D$ we have exactly one element $f(x) \in \mathbb{R}$.
The set D is called the domain of f [Definitionsbereich].
We denote by $f(D) := \{y \in Y \mid \text{there is a } x \in X \text{ with } f(x) = y\}$
the range of f [Bild]. The graph of f [Graph]
is the set $\Gamma = \{(x,y) \in D \times \mathbb{R} \mid y = f(x)\}$.

(b) Let $f, g: D \to \mathbb{R}$ be functions, $\lambda \in \mathbb{R}$. Then also

$$f+g: D \to \mathbb{R}, \; x \mapsto f(x)+g(x), \qquad \lambda f: D \to \mathbb{R}, \; x \mapsto \lambda f(x),$$

$$fg: D \to \mathbb{R}, \; x \mapsto f(x)g(x), \qquad \frac{f}{g}: D \to \mathbb{R}, \; x \mapsto \frac{f(x)}{g(x)}$$

are functions (we require $g(x) \neq 0$ for all $x \in D$ for defining $\frac{f}{g}$). For $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ with $f(D) \subseteq E$ is $g \circ f: D \to \mathbb{R}, \; x \mapsto g(f(x))$ the composition of f and g [Komposition].

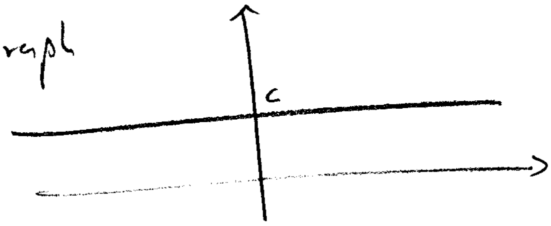
(c) We denote by $\mathrm{id}: D \to \mathbb{R}, \; x \mapsto x$ the identical map [identische Funktion]. If $g: f(D) \to \mathbb{R}$ is such that $g \circ f: D \to \mathbb{R}$ equals $\mathrm{id}: D \to \mathbb{R}$, then g is called the inverse function of f [Umkehrabbildung].

1.2 Examples:

(a) The constant function [konstante Funktion] is given by

$$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto c$$

for a fixed $c \in \mathbb{R}$. It has range $f(\mathbb{R}) = \{c\}$ and graph



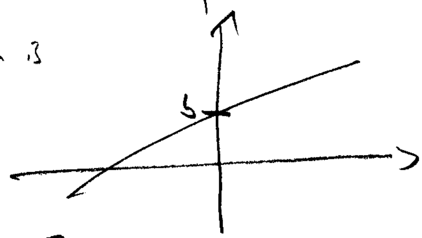
(b) The straight line [Gerade] is given by

$$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto ax+b$$

for $a, b \in \mathbb{R}$ with **slope** a [Steigung].

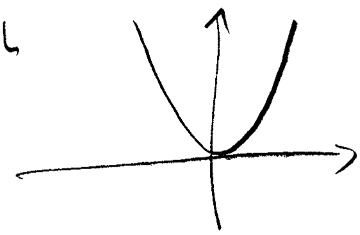
In the special case $a=1, b=0$ we have $f = \mathrm{id}$.

The range is $\underset{(a \neq 0)}{f(\mathbb{R}) = \mathbb{R}}$ and the graph is



(c) A polynomial [Polynom] is $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

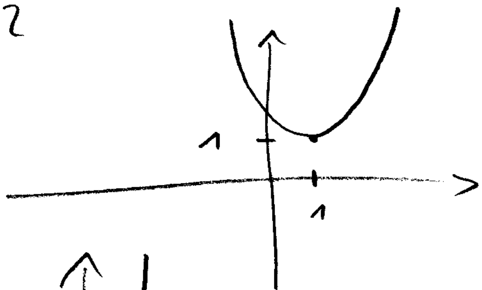
For $f(x) = x^2$ we have the graph



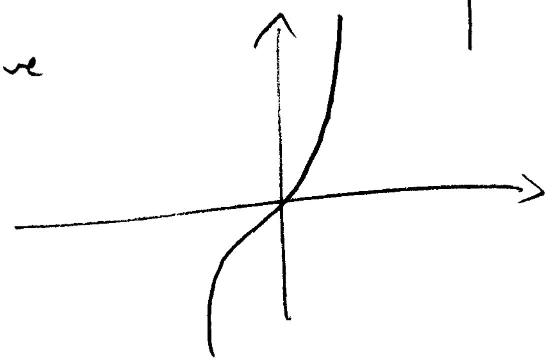
and the range $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$

For $f(x) = (x-1)^2 + 1 = x^2 - 2x + 2$

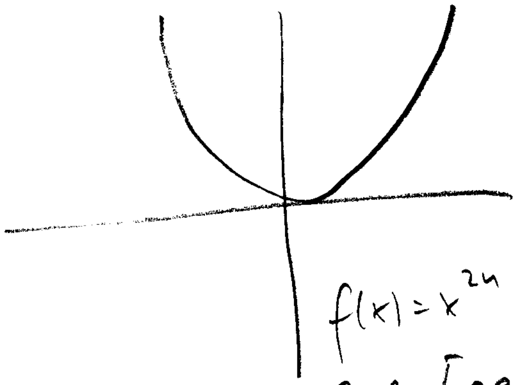
we have:
(shift of x^2)



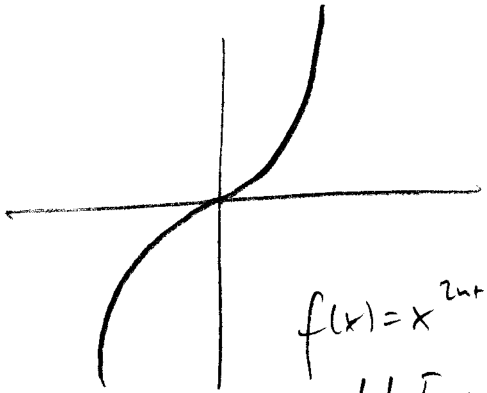
For $f(x) = x^3$ we have



More generally:



$f(x) = x^{2n}$
even [gerade]

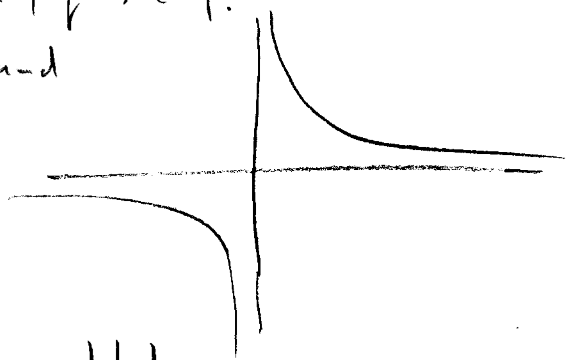


$f(x) = x^{2n+1}$
odd [ungerade]

(d) A rational function [rationale Funktion] is $f: D \longrightarrow \mathbb{R}$, $x \longmapsto \frac{p(x)}{q(x)}$

where p, q are polynomials. We might have restrictions on the domain, namely $D \subseteq \{x \in \mathbb{R} \mid q(x) \neq 0\}$.

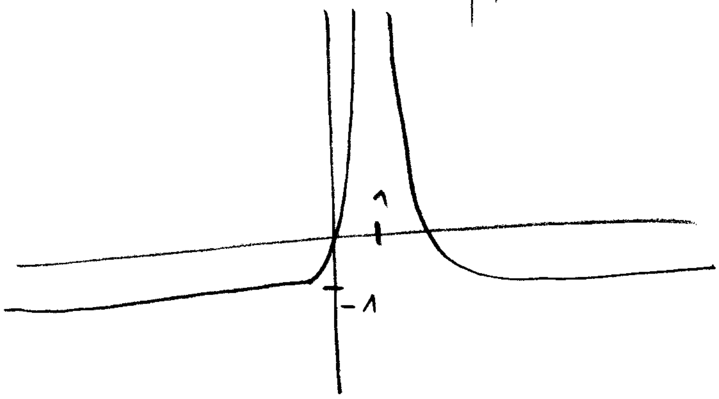
For $f(x) = \frac{1}{x} = x^{-1}$ we have $D \not\ni 0$ and

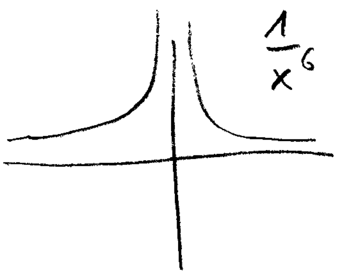


For $f(x) = (x-1)^{-6} - 1$

$1 \notin D$

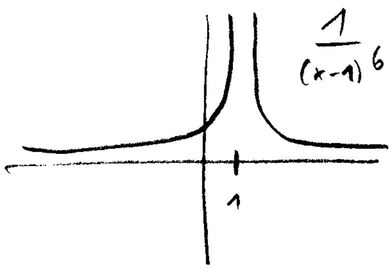
(shift of $\frac{1}{x^6}$)





$\frac{1}{x^6}$

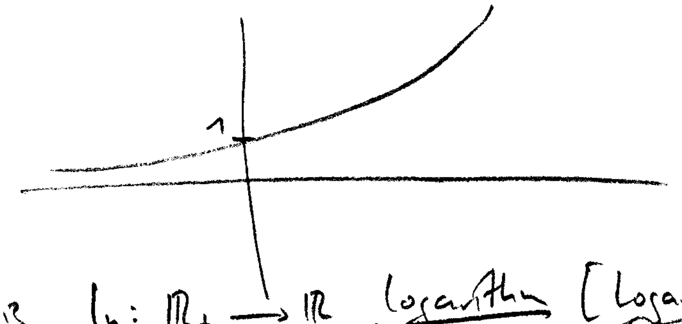
$\rightsquigarrow$



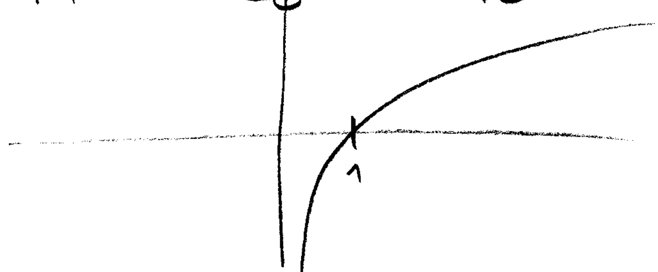
$\frac{1}{(x-1)^6}$

(e) $\exp: \mathbb{R} \to \mathbb{R}, \; x \mapsto e^x$ the exponential function [Exponentialfunktion].

Range $\mathbb{R}_+$, graph

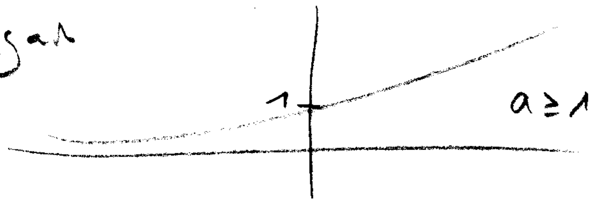


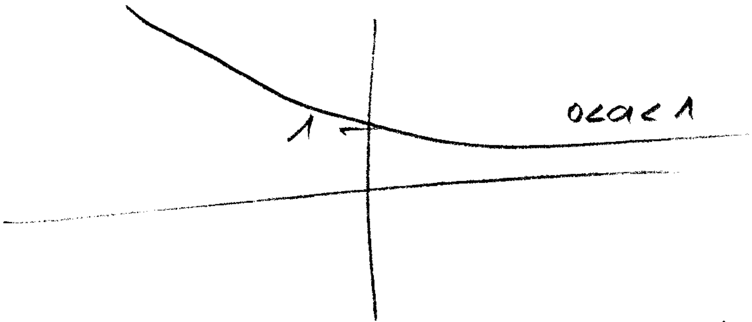
Its inverse function is $\ln: \mathbb{R}_+ \to \mathbb{R}$ logarithm [Logarithmus].



We also have $f: \mathbb{R} \to \mathbb{R}, \; x \mapsto a^x := e^{x \cdot \ln(a)}$, $a \in \mathbb{R}_+$.

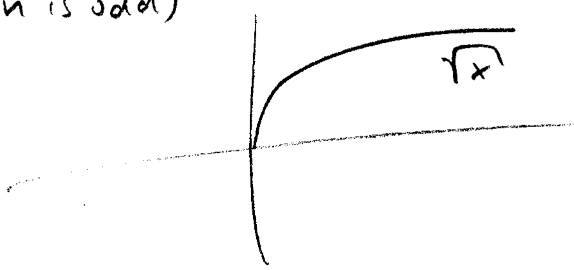
The graph is again (different slopes)





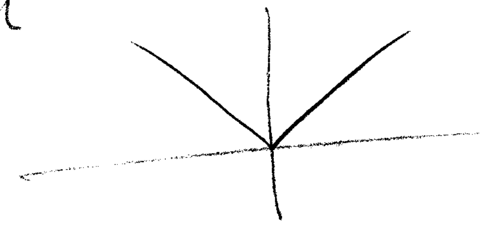
(f) The inverse function of x^n is the n-th square root [Wurzel]

$f: \mathbb{R}_+ \to \mathbb{R}, \; x \mapsto \sqrt[n]{x}$ (or $D = \mathbb{R}$ if n is odd)



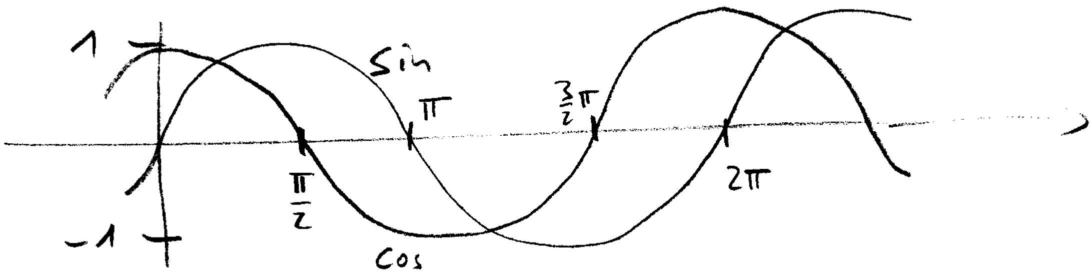
(g) The <u>absolute value</u> [Betragsfunktion] is $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$

with range $\mathbb{R}_+$ and graph



(h) We also have the trigonometric functions

$\sin: \mathbb{R} \to \mathbb{R}$, $\cos: \mathbb{R} \to \mathbb{R}$ <u>sine</u>, <u>cosine</u> [Sinus, Cosinus]



$\sin(-x) = -\sin(x)$

$\cos(-x) = \cos(x)$

$\sin^2(x) + \cos^2(x) = 1$

$\sin(x + 2\pi) = \sin(x)$

$\cos(x + 2\pi) = \cos(x)$

$\sin(x+\pi) = -\sin(x)$

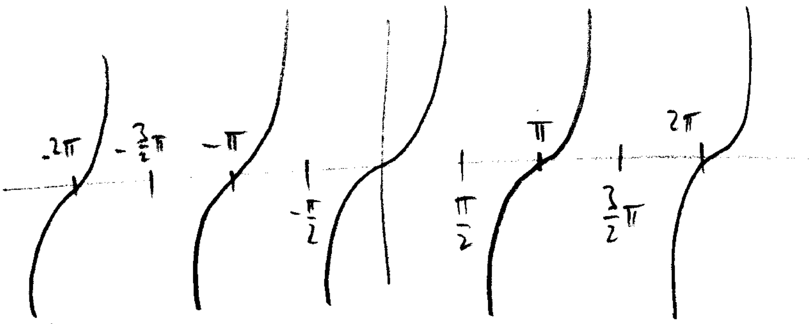
$\cos(x+\pi) = -\cos(x)$

$\sin(x + \frac{\pi}{2}) = \cos(x)$

$\cos(x+\frac{\pi}{2}) = -\sin(x)$

$\tan: \mathbb{R} \to \mathbb{R}$

$x \mapsto \frac{\sin(x)}{\cos(x)}$

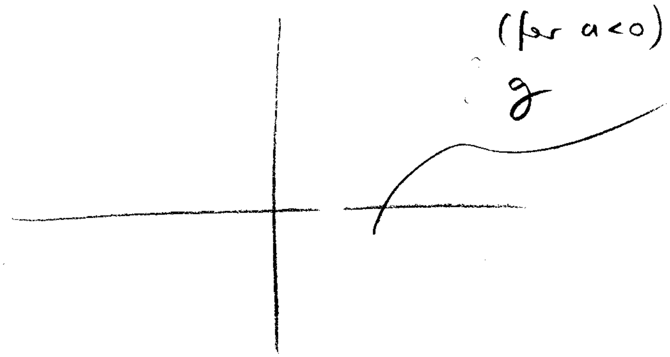


$k\pi + \frac{\pi}{2} \notin D$ for $k \in \mathbb{Z}$

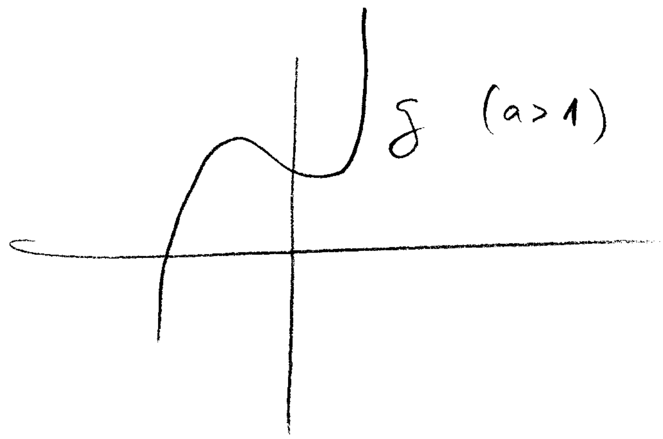
1.3 Def: If $f: \mathbb{R} \to \mathbb{R}$ is a function, then for $a \in \mathbb{R}$

(a) $g: \mathbb{R} \to \mathbb{R}$, $g(x) := f(x) + a$ is an up/down shift (y-axis)

(b) $g: \mathbb{R} \to \mathbb{R}$, $g(x) := f(x+a)$ is a left-right shift (x-axis)



(c) $g: \mathbb{R} \to \mathbb{R}$, $g(x) := a f(x)$ is a compression on the y-axis



(d) $g: \mathbb{R} \to \mathbb{R}$, $g(x) := f(ax)$ is a compression on the x-axis