§ 2 Sequences, asymptotics

2.1 Def: A sequence [Folge] $(a_n)_{n\in\mathbb{N}}$ is given by real numbers $a_n \in \mathbb{R}$ for $n \in \mathbb{N}$.
It converges [konvergiert] to $a \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.
Then, $a \in \mathbb{R}$ is the limit of $(a_n)_{n\in\mathbb{N}}$ [Grenzwert], and we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ for $n \to \infty$.
We write $\lim_{n\to\infty} a_n = \infty$, if for any $M \in \mathbb{N}$ there is a $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$. Likewise $\lim_{n\to\infty} a_n = -\infty$.
In this case $(a_n)_{n\in\mathbb{N}}$ converges to infinity (minus infinity) [Unendlich]. If $(a_n)_{n\in\mathbb{N}}$ does not converge, it is divergent [divergent].

2.2 Example: (a) $a_n \equiv c$. Converges to c. (constant sequence)

(b) $a_n = \frac{1}{n}$. Converges to 0.

[Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon}$. For $n \geq N$: $|a_n - 0| = |\frac{1}{n}| \leq \frac{1}{N} < \varepsilon$]

(c) $a_n = n$. Converges to ∞.

(d) $a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ diverges

⌈ Suppose $\lim_{n\to\infty} a_n = a$. Then, for $\varepsilon = \frac{1}{10} > 0$, there is a $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for $n \geq N$.
For n even: $|a| = |0 - a| < \frac{1}{10}$
n odd: $|1 - a| < \frac{1}{10}$ ⌋ Contradiction.

2.3 Rules: Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be convergent with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then we have:

(a) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(b) $\lim (a_n - b_n) = a - b$

(c) $\lim (\lambda a_n) = \lambda a$, $\lambda \in \mathbb{R}$

(d) $\lim (a_n b_n) = ab$

(e) $\lim \frac{a_n}{b_n} = \frac{a}{b}$, if $b_n \neq 0$ for all n and $b \neq 0$

(f) $\lim \sqrt{a_n} = \sqrt{a}$, if all $a_n \geq 0$ and $a \geq 0$

(g) $\lim a_n = \infty\ [-\infty]$, $\lim b_n = \infty\ [-\infty]$ $\Rightarrow$ $\lim a_n + b_n = \infty\ [-\infty]$

(h) $\lim a_n = \infty$, $\lim b_n = b \neq 0$ $\Rightarrow$ $\lim a_n b_n = \begin{cases} \infty & b > 0 \\ -\infty & b < 0 \end{cases}$

(i) $\lim a_n = \infty\ [-\infty] \Leftrightarrow \lim \frac{1}{a_n} = 0$

2.4 Example: (a) $a_n = \dfrac{n+5}{2n} = \dfrac{n(1+\frac{5}{n})}{n \cdot 2} = \dfrac{1+\frac{5}{n}}{2} \longrightarrow \frac{1}{2}$ for $n \to \infty$

because: $b_n :\equiv 1 \longrightarrow 1$ (by 2.2(a))

$c_n := 5 \cdot \frac{1}{n} \longrightarrow 0$ (by 2.2(b) and 2.3(d))

$d_n := 1 + \frac{5}{n} \longrightarrow 1$ (by 2.3(a))

$a_n = \frac{d_n}{2} \longrightarrow \frac{1}{2}$ (by 2.3(e))

(b) $$a_n = \frac{-n^3 + 3n - 5}{2n^2 + 3} = \underbrace{n}_{\to\infty} \cdot \underbrace{\frac{-1 + \frac{3}{n^2} - \frac{5}{n^3}}{2 + \frac{3}{n^2}}}_{\to -\frac{1}{2}} \longrightarrow -\infty \quad \text{(by 2.3(h))}$$

2.4 Def.: If $f: D \to \mathbb{R}$ is a function ($D \subseteq \mathbb{R}$) and $a \in D$, we write $\lim_{x \to a} f(x) = c$, if for every sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in D$ and $x_n \to a$ we have: $f(x_n) \to c$.
We may also define it for $a = \infty$ or $a = -\infty$ or $a \notin D$ in certain cases.

2.5 Example: (a) $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0$

since if $x_n \to \infty$, then $\frac{1}{x_n} \to 0$ by 2.3 (i).

(b) $f(x) = \frac{x+5}{2x}$, $\lim_{x \to \infty} \frac{x+5}{2x} = \frac{1 + \frac{5}{x}}{2} = \frac{1}{2}$

since $\lim_{x \to \infty} \frac{5}{x} = 5 \cdot \lim_{x \to \infty} \frac{1}{x} = 5 \cdot 0 = 0$.

(c) $f(x) = \frac{-x^3 + 3x - 5}{2x^2 + 3}$. $\lim_{x \to \infty} f(x) = -\infty$

(d) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 2} = \frac{1^2 - 3 \cdot 1 + 2}{1 - 2} = \frac{0}{-1} = 0$

$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = ?$ Problematic ... (see next chapter) use L'Hospital

2.6 Def.: $f: D \to \mathbb{R}$ is continuous, if $\lim_{x \to a} f(x) = f(a)$ for $a \in D$.

2.7 Examples: (a) Polynomials are continuous.
(b) $f(x) = x^a$ and $f(x) = a^x$ are continuous for $a > 0$
(c) ln, exp, sin, cos are continuous.

2.8 Example: (a) $\lim_{x \to 2} (3 + x^2 \ln(x)) = 3 + 4 \ln(2)$

(b) $\lim_{x \to \infty} e^{\frac{1}{x}} = \lim_{x \to 0} e^x = 1$

2.9 Remark: (a) $\lim_{x\to\infty} x^k e^{-x} = 0$ for all $k\in\mathbb{N}$.

Thus $\lim_{x\to\infty} x^2 e^{-x} = 0$. Also $\lim_{x\to\infty} (x^2+3x^3)e^{-x^2} = 0$

Since $\lim_{x\to\infty} x^2 e^{-x^2} = \lim_{x\to\infty} x e^{-x} = 0$

and $0 \le \lim_{x\to\infty} x^3 e^{-x^2} \le \lim_{x\to\infty} x^4 e^{-x^2} = 0$

(used: $a_n \ge b$ for all $n\in\mathbb{N}$, then also $\lim_{n\to\infty} a_n \ge b$)

(b) $\lim_{x\to\infty} x^{-a} \ln x = 0$ for $a>0$.

$\lim_{x\to 0} (3 + x\ln(x^4)) = ?$

$x \to \infty \Rightarrow e^{-x} \to 0 \Rightarrow (e^{-x})\underbrace{\ln((e^{-x})^4)}_{=(-4x)} \to 0$