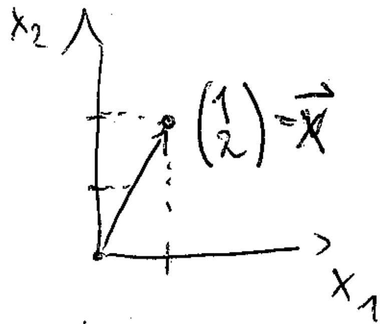
Vektorrechnung

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad x_i \in \mathbb{R}$$

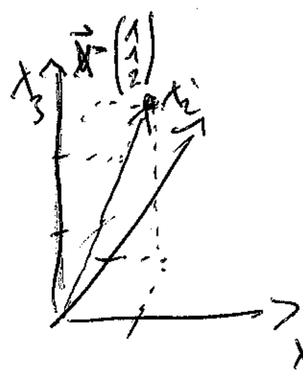
Vektor / n-Tupel

vector / n-tuple

$\mathbb{R}^2$ Ebene plane



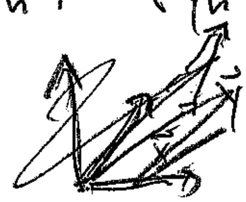
$\mathbb{R}^3$ Raum space

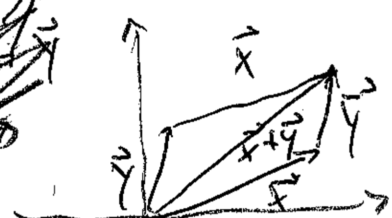


Vektoraddition

vector addition

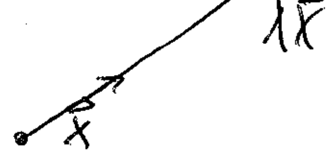
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$



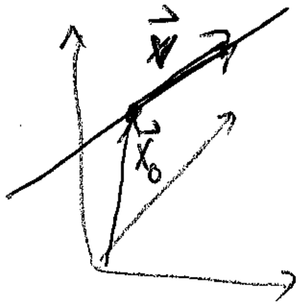


Skalarmultiplikation scalar multiplication

$$\underset{\in \mathbb{R}}{\lambda} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$



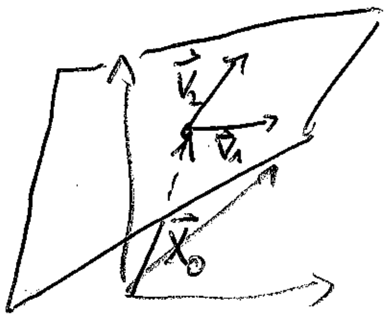
Geradengleichung
Equation of a line



$$\vec{x}(\lambda) = \vec{x}_0 + \lambda \vec{v}$$

Aufpunkt / base point ($\vec{x}_0$), Parameter / parameter (λ), Richtun[gs]vekt[or] / dire[ction] vec[tor] ($\vec{v}$)

Ebenengleichung
Equation of a plane

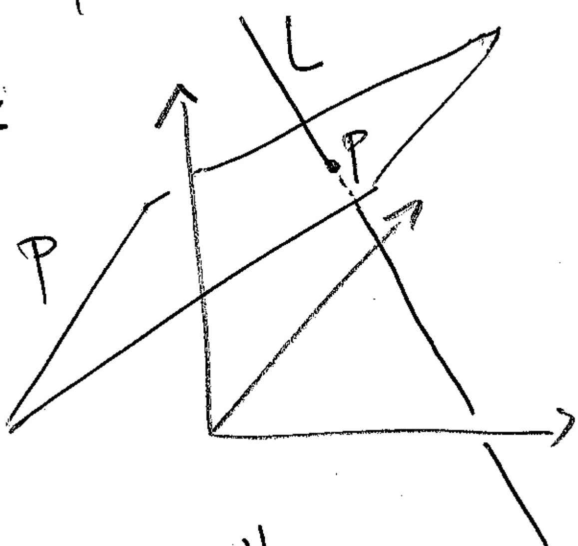


$$\vec{x}_{(\lambda_1,\lambda_2)} = \vec{x}_0 + \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2$$

2 Parameter (λ_1, λ_2)

⋮

Schnittpunkte intersection points



$$P \cap L = \{p\}$$

$$p = ?$$

⇝ lineares Gleichungssystem
linear sys. of equat[ions]

Bsp: $P = \left\{ \vec{x}^{(\lambda_1,\lambda_2)} = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

$L = \left\{ \vec{x}_{(\mu)} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \right\}$

Search with
Suchen $\lambda_1, \lambda_2, \mu$ mit $\vec{x}(\lambda_1, \lambda_2) = \vec{x}(\mu)$

$$\Rightarrow \begin{aligned} 2\lambda_2 + \mu &= 4 \\ \lambda_1 \quad + \mu &= 1 \\ \lambda_2 - 3\mu &= -2 \end{aligned}$$

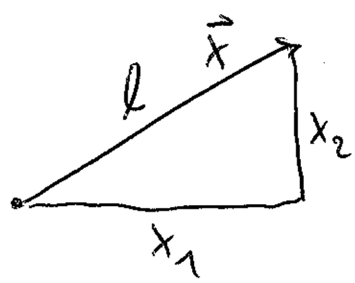
Lösen!

Länge / Norm ~~Abstand~~

Length ~~distance~~

$$\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Idee: Satz v. Phythagoras

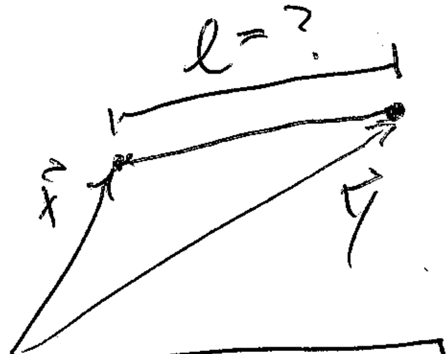


$$\ell^2 = x_1^2 + x_2^2$$

Abstand distance

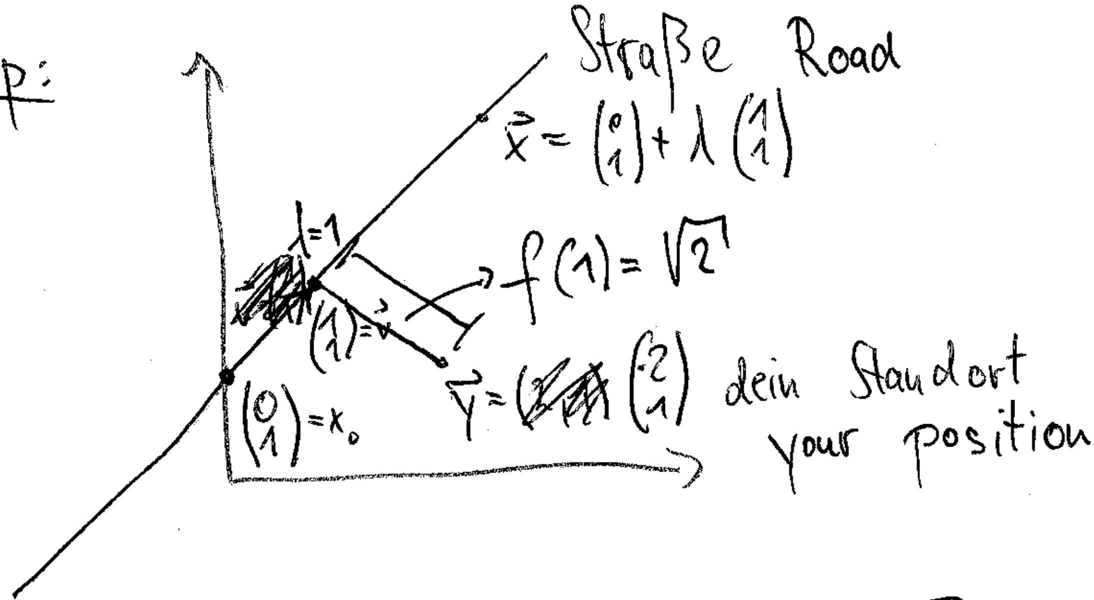
zwischen $\vec{x}$ und $\vec{y}$

between



$$d(\vec{x}, \vec{y}) = \|\vec{y} - \vec{x}\| = \sqrt{(y_1 - x_1)^2 + \ldots + (y_n - x_n)^2}$$

Bsp:



Frage: Entfernung zu Straße?
Distance to Road?

Antwort:

$$f(\lambda) = d(\vec{x}(\lambda), \vec{y}) = \left\| \begin{pmatrix} \lambda - 2 \\ 1 + \lambda - 1 \end{pmatrix} \right\| = \sqrt{(\lambda-2)^2 + \lambda^2} = \sqrt{\underbrace{2\lambda^2 - 4\lambda + 4}_{g(\lambda)}}$$

Suchen Minimum von $f(\lambda)$ $\hat{=}$ Minimum von $g(\lambda)$
Search " of " " of

Ableitung ~~g(λ)~~ $= g'(\lambda) = 4\lambda - 4 \overset{!}{=} 0$
derivative

$$\Rightarrow \lambda = 1$$

$$\Rightarrow f(1) = \sqrt{2} \;\hat{=}\; \text{Entfernung zur Straße}$$

oder mit

Skalarprodukt

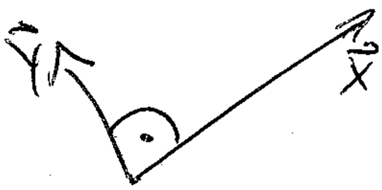
Scalar product

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \ldots + x_n y_n \in \mathbb{R}$$

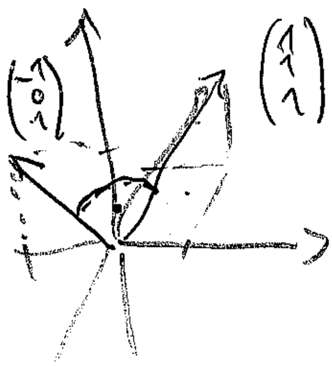
Spezialfälle:

1) $\vec{x} = \vec{y}$: $\quad \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$

2) $\vec{x} \perp \vec{y}$, $\vec{x}$ und $\vec{y}$ sind orthogonal / are orthogonal



$$\vec{x} \cdot \vec{y} = 0$$

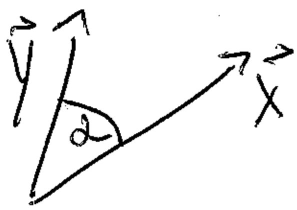


$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1 + 1 = 0$$

$\Leftrightarrow$

allgemein:

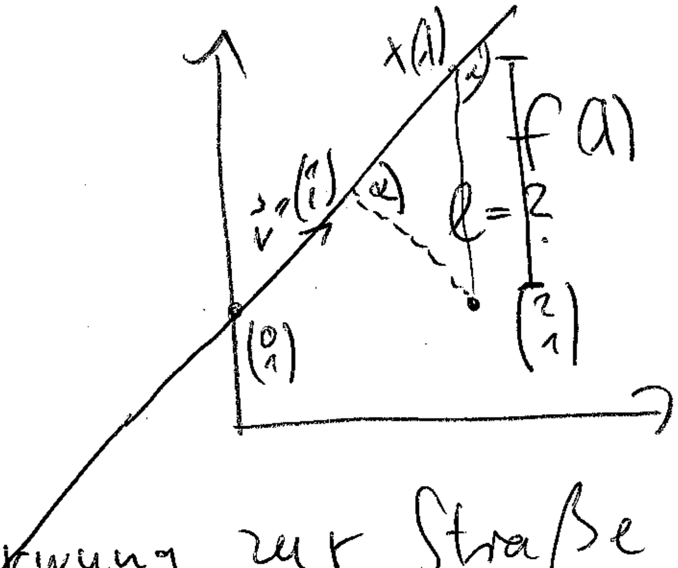
$$\vec{x} \cdot \vec{y} = \cos(\alpha) \cdot \|\vec{x}\| \cdot \|\vec{y}\|$$



1) $\alpha = 0°$. $\quad \cos(\alpha) = 1$

2) $\alpha = 90°$ $\quad \cos(\alpha) = 0$

nochmal Bsp:

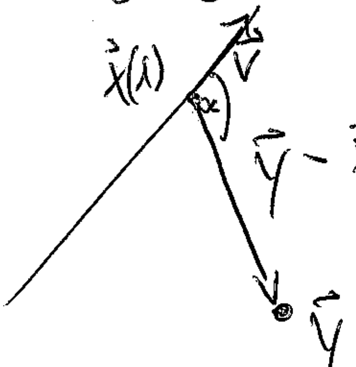


Suche $\ell \mathrel{\hat{=}}$ Entfernung zur Straße

Idee: Betrachte Winkel α.

Idea: $f(\lambda)$ minimal $\Longleftrightarrow$ $\alpha = 90°$

$\Rightarrow$ Suchen λ mit $\vec{y} - \vec{x}(\lambda) \perp \vec{v}$



$$(\vec{y} - \vec{x}(\lambda)) \cdot \vec{v} = \begin{pmatrix} 2-\lambda \\ -\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 - \lambda - \lambda$$

$$= 2 - 2\lambda \overset{!}{=} 0$$

$$\Rightarrow \lambda = 1 \ \ldots$$

Kreuzprodukt (NUR IM $\mathbb{R}^3$)

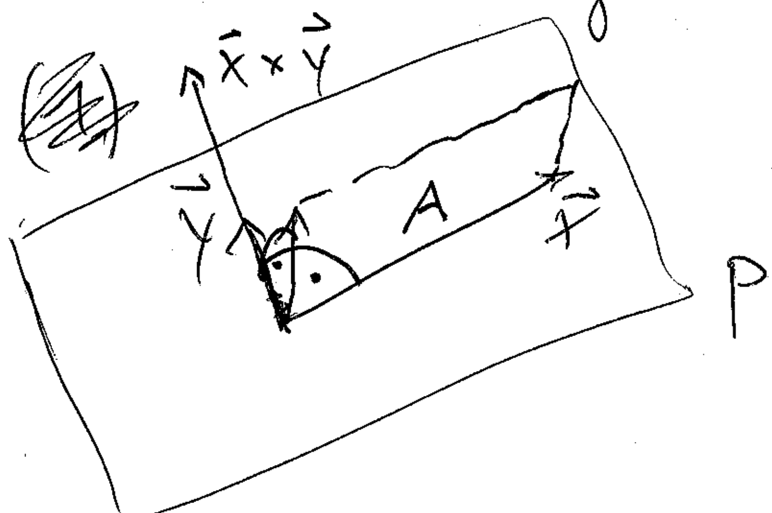
cross product ONLY IN $\mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Bsp $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 0 \\ -1 - 1 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

geometrische Bedeutung:

geometrical meaning



$A = \|\vec{x} \times \vec{y}\|$

1) $\vec{x} \times \vec{y}$ ist orthogonal zu $P = \{\lambda_1 \vec{x} + \lambda_2 \vec{y}\}$

2) $\|\vec{x} \times \vec{y}\|$ = Fläche des Parallelograms
area of parallelogram

Beweis: Proof

nur ein Beispiel:

$(\vec{x} \times \vec{y}) \perp \vec{x}$ denn:

$$\vec{x}\,(\vec{x} \times \vec{y}) = x_1 x_2 y_3 - x_1 x_3 y_2 + x_2 x_3 y_1 - x_2 x_1 y_3 + x_3 x_1 y_2 - x_3 x_2 y_1 = 0.$$