Kreuzprodukt (NUR IM $\mathbb{R}^3$) 6

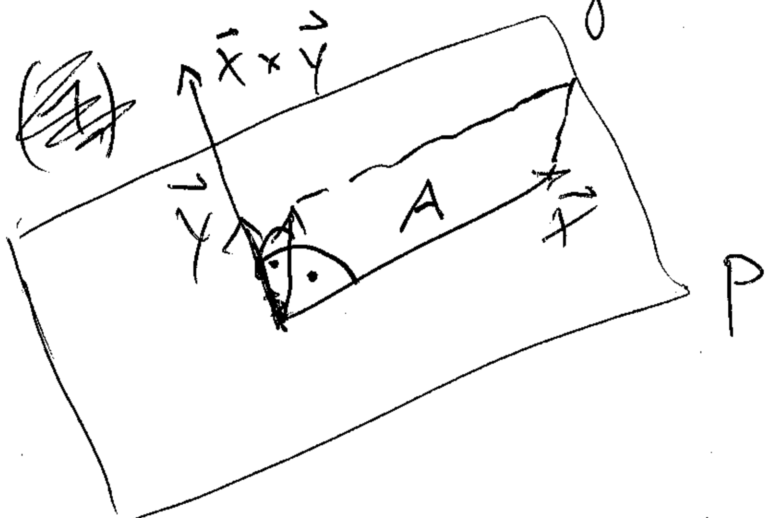
cross product ONLY IN $\mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Bsp $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 0 \\ -1 - 1 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

geometrische Bedeutung:

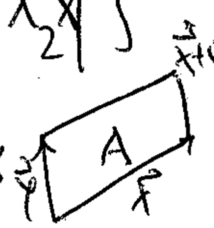
geometrical meaning



$A = \|\vec{x} \times \vec{y}\|$

1) $\vec{x} \times \vec{y}$ ist orthogonal zu $P = \{\lambda_1 \vec{x} + \lambda_2 \vec{y}\}$

2) $\|\vec{x} \times \vec{y}\|$ = Fläche des Parallelograms
area of parallelogram



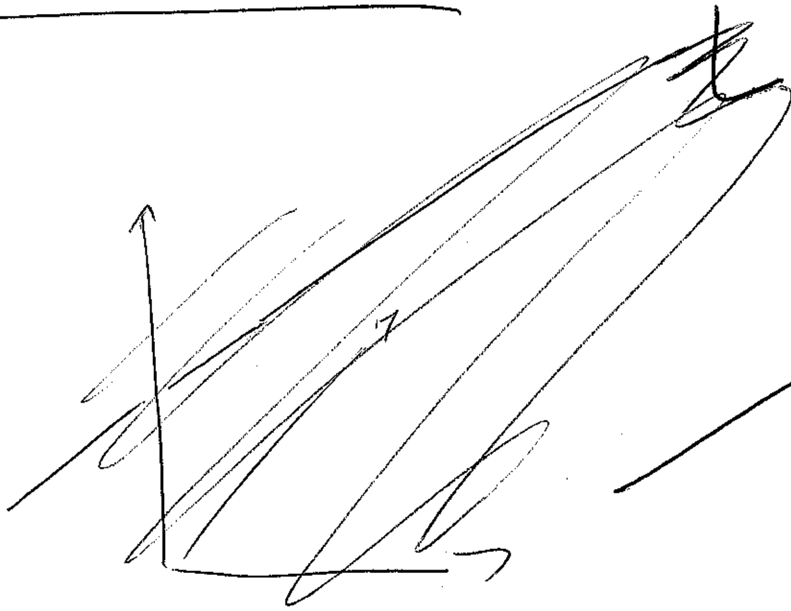
Beweis: Proof

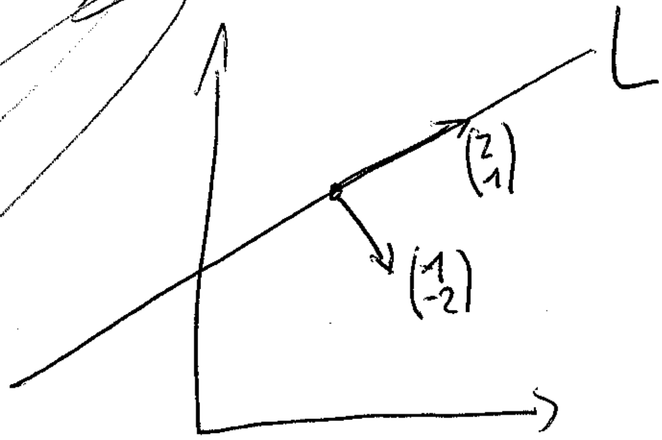
nur ein Beispiel:

$(\vec{x} \times \vec{y}) \perp \vec{x}$ denn:

$$\vec{x}\,(\vec{x} \times \vec{y}) = \cancel{x_1 x_2 y_3} - \cancel{x_1 x_3 y_2} + \cancel{x_2 x_3}\, y_1 - \cancel{x_2 x_1 y_3} + \cancel{x_3 x_1 y_2} - \cancel{x_3 x_2}\, y_1 = 0.$$

Geraden beschreiben — How to describe lines?





explizit / **Parameterdarstellung**

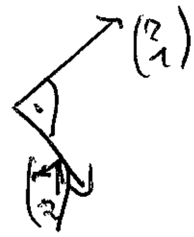
explicit / parametrization

$$L = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{R} \right\}$$

implizit / **Gleichung**

implicit / equation

$$L = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \underbrace{x_1 - 2x_2}_{= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = -4 \right\}$$



$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ Richtungsvektor — direction vector

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Normalenvektor — normal vector

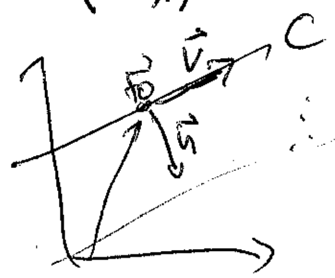
(ebenso: $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, … $-1x_1 + x_2 = 4$.)

<u>Umrechnung:</u> $L = \{\vec{x}(\lambda) \mid \lambda \in \mathbb{R}\} = \{\vec{x} \mid \vec{n}\vec{x} = c\}$

(A) $\vec{x}(\lambda) = \vec{x}_0 + \lambda \vec{v} \rightsquigarrow \vec{n} \cdot \vec{x} = c$, $c \in \mathbb{R}$

1) Finde $\vec{n}$. z.B. $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightsquigarrow \vec{n} = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$

2) Finde c. z.B. $c = \vec{n}\,\vec{x}_0$



<u>Bsp:</u> $\vec{x}(\lambda) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$\vec{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad c = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 13$

$\cancel{L = \{\vec{x}(\lambda) \mid \lambda \in \mathbb{R}\} = \{\vec{x} \mid \vec{n} \cdot}$

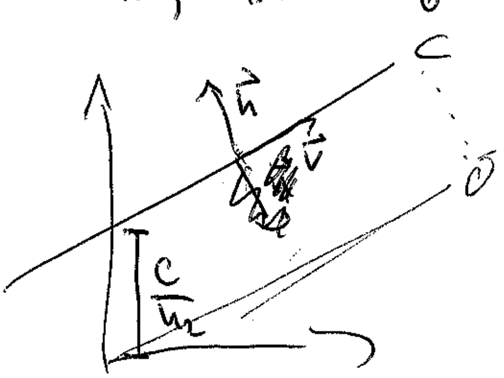
$L = \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \end{pmatrix} \mid \lambda \in \mathbb{R}\right\} = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid 3x_1 + 4x_2 = 13\right\}$

(B) $\vec{n} \cdot \vec{x} = c \rightsquigarrow \vec{x}(\lambda) = \vec{x}_0 + \lambda \vec{v}$.

1) Finde $\vec{v}$. zB $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \rightsquigarrow \vec{v} = \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix}$

2) Finde $\vec{x}_0$. z.B. falls $n_2 \neq 0$

$\vec{x}_0 = \begin{pmatrix} 0 \\ \frac{c}{n_2} \end{pmatrix}$ $\quad$ (sonst $\vec{x}_0 = \begin{pmatrix} \frac{c}{n_1} \\ 0 \end{pmatrix}$)

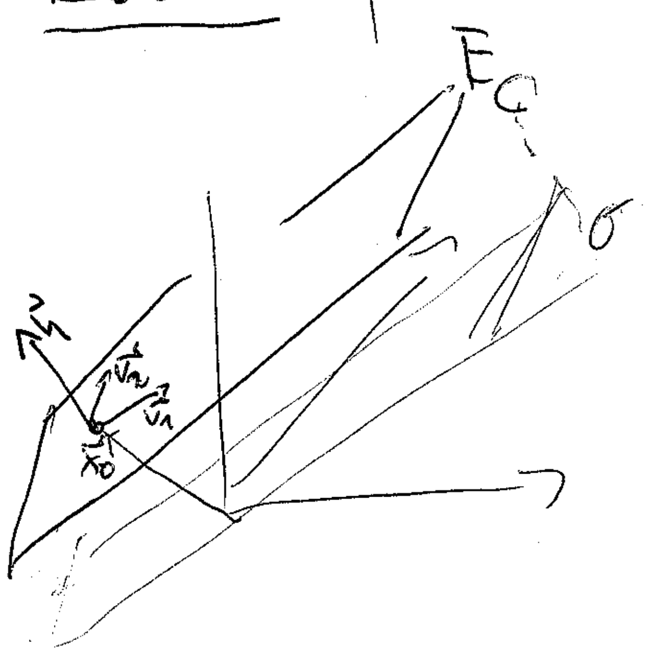


Bsp: $3x_1 + 4x_2 = 13$

$$\vec{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad \vec{x}_0 = \begin{pmatrix} 0 \\ \frac{13}{4} \end{pmatrix}$$

Check: $\begin{pmatrix} 4 \\ -3 \end{pmatrix} = -\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ ✓ $\quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{13}{4} \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ ✓

Ebenen planes



$$E = \{ \vec{x}_0 + \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \mid \lambda_1, \lambda_2 \in \mathbb{R} \} \quad \text{explizit}$$

$$= \{ \vec{x} \mid \vec{n} \cdot \vec{x} = c \} \quad \text{implizit}$$

$$n_1 x_1 + n_2 x_2 + n_3 x_3$$

(A) $\vec{x}(\lambda_1, \lambda_2) = \vec{x}_0 + \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \rightsquigarrow \vec{n}\vec{x} = c$

1. Finde $\vec{n}$.

Idee: Falls $\vec{x} \in E$, ~~dann~~ auch $\vec{x} + \vec{v}_1$ und $\vec{x} + \vec{v}_2 \in E$

$$\Rightarrow \vec{n}\vec{x} = c = \vec{n}(\vec{x} + \vec{v}_1) = c = \vec{n}(\vec{x} + \vec{v}_2).$$

$$\Rightarrow \vec{n}\vec{v}_1 = 0 = \vec{n}\vec{v}_2$$

Suchen $\vec{n}$ mit $\vec{n} \perp \vec{v}_1$ und $\vec{n} \perp \vec{v}_2$.

Variante a) $\vec{n}\vec{v}_1 = v_1^1 n_1 + v_1^2 n_2 + v_1^3 n_3 \overset{!}{=} 0$

$\vec{n}\vec{v}_2 = v_2^1 n_1 + v_2^2 n_2 + v_2^3 n_3 \overset{!}{=} 0$

LGS $\xrightarrow{GA}$ Lösen nach n_1, n_2, n_3.

Variante b) $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

2. Finde c.

$$c = \vec{n}\vec{x}_0.$$

Bsp: $\vec{x}(\lambda_1,\lambda_2) = \begin{pmatrix}3\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_2\begin{pmatrix}-1\\1\\1\end{pmatrix}$

1. a) $\left(\begin{array}{ccc|c}1&2&3&0\\-1&1&1&0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1&2&3&0\\0&3&4&0\end{array}\right)$

n_3 frei wählbar z.B. $n_3 = 1$

$3n_2 + 4n_3 = 0 \Rightarrow n_2 = -\frac{4}{3}$

$n_1 + 2n_2 + 3n_3 \Rightarrow n_1 = -\frac{9}{3} + \frac{8}{3} = -\frac{1}{3}$

$\Rightarrow \vec{n} = \frac{1}{3}\begin{pmatrix}-1\\-4\\3\end{pmatrix}$

b) $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-4\\3\end{pmatrix}$

2. mit a) $c = \vec{n}\vec{x}_0 = \frac{1}{3}(-3-4) = -\frac{7}{3}$

b) $c = -7$.

(B) $\vec{n}\vec{x} = c \rightsquigarrow \vec{x}(\lambda_1,\lambda_2) = \ldots$

1. Finde $\vec{v}_1, \vec{v}_2$.

Wir suchen Vektoren mit $\vec{n}\vec{v}_i = 0$.

$\vec{n} = \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$. Falls $n_1 \neq 0$, nehmen wir $\vec{v}_1 = \begin{pmatrix}n_2\\-n_1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}n_3\\0\\-n_1\end{pmatrix}$

(Sonst analog für $n_2 \neq 0$ oder $n_3 \neq 0$)

2. Finde $\vec{x}_0$

Falls $n_1 \neq 0$, nehmen wir $\vec{x}_0 = \begin{pmatrix}\frac{c}{n_1}\\0\\0\end{pmatrix}$.

Bsp: $\cancel{\begin{pmatrix}0\\2\\3\end{pmatrix}} \quad \begin{pmatrix}0\\2\\3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 4$

1. $n_2 \neq 0 \Rightarrow \quad \vec{v}_1 = \cancel{\begin{pmatrix}-2\\0\\0\end{pmatrix}} = \begin{pmatrix}-n_2\\n_1\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\0\end{pmatrix}$

$$\vec{v}_2 = \begin{pmatrix}\cancel{-n_2}\\ n_3\\ -n_2\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix}$$

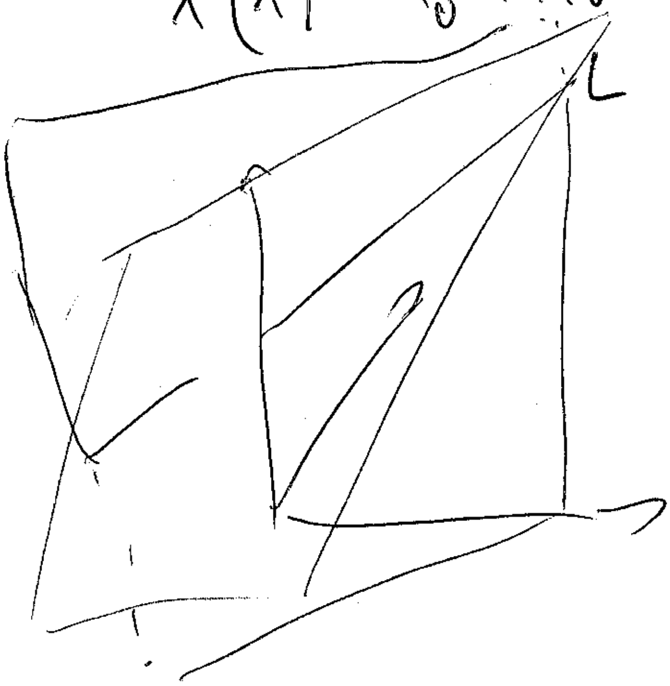
2. $\vec{x}_0 = \begin{pmatrix}0\\ \frac{4}{2}\\ 0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$

Geraden im Raum ??

2 Gleichungen

$\vec{x}(\lambda) = \vec{x}_0 + \lambda \vec{v} \rightsquigarrow$ $\vec{n}_1 \vec{x} = c_1$

$\vec{n}_2 \vec{x} = c_2$



so wie eben / as before

Idee: $\vec{v} \perp \vec{n}_1$, $\vec{v} \perp \vec{n}_2$

$\rightsquigarrow$ ~~lineare Gl~~ LGS

allgemein: Unterraum im $\mathbb{R}^n$

$\vec{x}(\lambda_1, \dots, \lambda_k) = \vec{x}_0 + \lambda_1 \vec{v}_1 + \dots + \lambda_k \vec{v}_k \rightsquigarrow$ $\vec{n}_1 \vec{x} = c_1$ $\quad \vec{n}_i \in \mathbb{R}^n$

$\vdots$

$\vec{x}_0, \vec{v}_i \in \mathbb{R}^n$ $\quad \vec{n}_{n-k} \vec{x} = c_{n-k}$

Dimension $k \rightsquigarrow n-k$ Gleichungen

Idee: $\vec{v}_i \perp \vec{n}_j$ für alle i, j

$\rightsquigarrow$ LGS. Lösen mit GA.