UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Non-Commutative (Algebraic) Geometry
Winter term 2017/18

Assignment 1
Hand in on Tuesday, 02.01.18, Mailbox 014 (basement of E2.5)

Exercise 1.
Consider the ideal given by

$$I = \langle x^4 + y^4 + 2x^2y^2 - x^2 - y^2 \rangle \subset \mathbb{R}[x, y].$$

(i) Compute the vanishing locus $V(I)$ and draw a picture of it.

(ii) Is I a prime? Is it radical, i.e. $\sqrt{I} = I$?

(iii) Does Hilbert's Nullstellensatz hold for I? (Even though $\mathbb{R}$ is not algebraically closed)

Exercise 2.
From Amitsur's Nullstellensatz for matrices we know that 1 must be in the ideal which is generated by $x_1x_2 - x_2x_1 - 1$ and the ideal m_2 of polynomial identities for 2×2-matrices. The latter is generated by $[[x_1, x_2]^2, x_3]$ and $s_4(x_1, x_2, x_3, x_4)$ (and substitution of variables). Find an explicit representation of 1 in terms of those generators.

Exercise 3 (Prove the claim in Remark 3.7.)**.**
Let c be a homogenous central polynomial, i.e. $c(A) \in \mathbb{C} \cdot 1_d$ for all $a \in M_d(A)$. Show that for $f_1 = c$ and $f_2 = 1 + c^2$ we have:

- $\mathrm{Tr}(f_1(X)) = 0 = \mathrm{Tr}(f_2(X))$ has no solution $X \in M_d(\mathbb{C})$;
- but there are no $\lambda_1, \lambda_2 \in \mathbb{C}$, s.t.

  $$\lambda_1 f_1 + \lambda_2 f_2 \sim 1 + p,$$

  where p a polynomial identity in $M_d(\mathbb{C})$.

Exercise 4 (Prove part 2 of Lemma 3.8)**.**
Let $n \geq 2$ and $g, g_1 \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ such that

$$gpg_1 = g_1pg \qquad \forall p \in \mathbb{C}\langle x_1, \dots, x_n \rangle.$$

Show that g and g_1 are linearly dependent.

bitte wenden

Exercise 5.
Prove by algebraic manipulations the rational identity in $\mathbb{C}\langle\!\langle x, y, z \rangle\!\rangle$:

$$y^{-1} + y^{-1}(z^{-1}x^{-1} - y^{-1})^{-1}y^{-1} = (y - zx)^{-1}.$$

Exercise 6.
Let A be a unital algebra, $\Omega^n(A)$ the algebra of non-commutative n-forms, $\omega \in \Omega^n(A)$ and $\eta \in \Omega^m(A)$. Show that for boundary operator d the graded Leibniz rule holds:

$$\mathrm{d}(\omega\eta) = \mathrm{d}\omega \cdot \eta + (-1)^n \omega \mathrm{d}\eta.$$