UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Tobias Mai

Assignments for the lecture
Introduction to Noncommutative Differential Geometry
Summer term 2019

A brief reminder on the theory of unbounded linear operators

Let $(\mathcal{H}_1, \langle \cdot, \cdot \rangle)$ and $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)$ be complex Hilbert spaces; the norms induced by the inner product are denoted by $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively.

1. The notion of unbounded linear operators

By an *unbounded (linear) operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$, we mean a linear map

$$T : \ \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$$

that is defined on a linear subspace $\operatorname{dom} T$ of $\mathcal{H}_1$, called the *domain of* T. We say that T is *densely defined* if $\operatorname{dom} T$ is dense in $\mathcal{H}_1$, i.e., if $\overline{\operatorname{dom} T}^{\|\cdot\|_1} = \mathcal{H}_1$
The *graph of* T, which we will denote by $G(T)$, is defined as

$$G(T) := \{(x, Tx) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid x \in \operatorname{dom} T\}.$$

It is thus linear subspace of the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ with the inner product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle_1 + \langle y_1, y_2 \rangle_2 \qquad \text{for } (x_1, y_1), (x_2, y_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Lemma 1. *A linear subspace $G \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ is the graph of an unbounded linear operator (i.e., there is an unbounded linear operator $T : \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ such that $G = G(T)$) if and only if $G \cap (\{0\} \times \mathcal{H}_2) = \{(0, 0)\}$.*

2. Closed and closable operators

Let $T : \ \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ be an unbounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.
An unbounded operator $S : \mathcal{H}_1 \supseteq \operatorname{dom} S \to \mathcal{H}_2$ is called an *extension of* T, written as $S \subseteq T$, if $G(T) \subseteq G(S)$ holds, i.e., if $\operatorname{dom} T \subseteq \operatorname{dom} S$ and $Sx = Tx$ for every $x \in \operatorname{dom} T$.
We say that the unbounded operator T is

- *closed*, if $G(T)$ is closed in $\mathcal{H}_1 \oplus \mathcal{H}_2$; explicitly, this means that for every sequence $(x_n)_{n=1}^\infty$ in $\operatorname{dom} T$ which converges in $\mathcal{H}_1$ to a point $x \in \mathcal{H}_1$ and for which $(Tx_n)_{n=1}^\infty$ is convergent in $\mathcal{H}_2$ to a point $y \in \mathcal{H}_2$, it holds true that $x \in \operatorname{dom} T$ and $y = Tx$.

- *closable*, if T admits an extension S that is closed.

Theorem 1. *For an unbounded operator $T : \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$, the following statements are equivalent:*

(i) *T is closable;*

(ii) *for every sequence $(x_n)_{n=1}^\infty$ in $\operatorname{dom} T$ which converges to 0 in $\mathcal{H}_1$ and for which $(Tx_n)_{n=1}^\infty$ converges in $\mathcal{H}_2$ to a point $y \in \mathcal{H}_2$, we necessarily have that $y = 0$;*

(iii) $\overline{G(T)} \cap (\{0\} \times \mathcal{H}_2) = \{(0,0)\}$.

It is worthwhile to take a closer look on the proof that (iii) implies (i). It follows from Lemma 1 that if (iii) holds, then $\overline{G(T)}$ must be the graph of an unbounded linear operator, say $\overline{T} : \mathcal{H}_1 \supseteq \operatorname{dom} \overline{T} \to \mathcal{H}_2$. The operator $\overline{T}$ is thus a closed extension of T; in fact, it is the (unique) minimal closed extension (i.e., for every other closed operator S that satisfies $T \subseteq S$, it follows that $\overline{T} \subseteq S$), called the *closure of* T. Furthermore, its domain $\operatorname{dom} \overline{T}$ is the closure of $\operatorname{dom} T$ with respect to the *graph norm* $\|\cdot\|_T$ which is defined by $\|x\|_T^2 := \|x\|_1^2 + \|Tx\|_2^2$ for each $x \in \operatorname{dom} T$.

3. The adjoint operator

Let now $T : \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ be a densely defined unbounded linear operator. For every $y \in \mathcal{H}_2$, we introduce a linear functional $\varphi_y : \operatorname{dom} T \to \mathbb{C}, x \mapsto \langle Tx, y\rangle_2$. Using this notation, we may define

$$\operatorname{dom} T^* := \{y \in \mathcal{H}_2 \mid \varphi_y \text{ is continuous on } \operatorname{dom} T \text{ with respect to } \|\cdot\|_1\},$$

which is clearly a subspace of $\mathcal{H}_2$. Since $\operatorname{dom} T$ is dense in $\mathcal{H}_1$, φ_y for every $y \in \operatorname{dom} T^*$ extends uniquely to a bounded linear functional $\widehat{\varphi_y}$ on $\mathcal{H}_1$; by the Riesz representation theorem, the latter must be of the form $\widehat{\varphi_y}(x) = \langle x, T^*y\rangle_1$ for all $x \in \mathcal{H}_1$ with a unique vector $T^*y \in \mathcal{H}_1$. The assignment $y \mapsto T^*y$ is in fact linear on $\operatorname{dom} T^*$, so that this construction results in an unbounded linear operator

$$T^* : \ \mathcal{H}_2 \supseteq \operatorname{dom} T^* \to \mathcal{H}_1,$$

called the *adjoint of* T.

Theorem 2. *Let $T : \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ be a densely defined unbounded linear operator.*

(i) *The adjoint operator T^* is always closed.*

(ii) *If T is closed, then T^* is densely defined and the operator $T^{**} := (T^*)^*$ satisfies $T^{**} = T$.*

(iii) *The operator T is closable if and only if its adjoint T^* is densely defined; in this case, we have that $T^{**} = \overline{T}$.*

4. Symmetric and selfadjoint operators

Throughout the following, let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. A densely defined operator $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ is called

- *symmetric*, if $T \subseteq T^*$, or in other words, if $\langle Tx_1, x_2 \rangle = \langle x_1, Tx_2 \rangle$ for all $x_1, x_2 \in \mathcal{H}$.
- *selfadjoint*, if $T = T^*$.
- *maximally symmetric*, if T is symmetric and if the following holds: whenever S is a symmetric extension of T, it follows that $S = T$.
- *essentially selfadjoint*, if T is symmetric with selfadjoint closure $\overline{T}$.

Lemma 2.

(i) Every symmetric operator is closable.

(ii) Every selfadjont operator is maximally symmetric.

(iii) A densely defined operator $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ is essentially selfadjoint if and only if $\overline{T} = T^*$.

Suppose that T is densely defined and symmetric. The *defect indices* $n_\pm(T) \in [0, \infty]$ of T are defined by

$$n_+(T) := \dim(T + i)^\perp = \dim\ker(T^* - i) \quad \text{and} \quad n_-(T) := \dim(T - i)^\perp = \dim\ker(T^* + i).$$

Theorem 3. *Let $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be densely defined and symmetric. Then the following statements are equivalent:*

(i) T is essentially selfadjoint;

(ii) $n_+(T) = n_-(T) = 0$;

(iii) $\operatorname{ran}(T + i)$ and $\operatorname{ran}(T - i)$ are dense.

Suppose in addition that T is closed. Then the following statements are equivalent:

(i) T is selfadjoint;

(ii) $n_+(T) = n_-(T) = 0$;

(iii) $\operatorname{ran}(T + i) = \operatorname{ran}(T - i) = \mathcal{H}$.

For closed operators, we actually have the following.

Theorem 4. *Let $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be densely defined, closed, and symmetric. Then we have the following:*

(i) T is selfadjoint if and only if $n_+(T) = n_-(T) = 0$.

(ii) T is maximally symmetric if and only if $n_+(T) = 0$ or $n_-(T) = 0$.

(iii) T has a selfadjoint extension if and only if $n_+(T) = n_-(T)$.

5. Resolvent set and spectrum

For any densely defined unbounded linear operator $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$, we define its *resolvent set* $\rho(T)$ by

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda 1) : \operatorname{dom} T \to \mathcal{H} \text{ is bijective and } (T - \lambda 1)^{-1} \in B(\mathcal{H})\}$$

and its *spectrum* $\sigma(T)$ by $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Lemma 3. *Suppose that $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ is densely defined and closed.*

(i) If $(T - \lambda 1) : \operatorname{dom} T \to \mathcal{H}$ is bijective for a $\lambda \in \mathbb{C}$, then its inverse $(T - \lambda 1)^{-1}$ is bounded.

(ii) The spectrum $\sigma(T) \subseteq \mathbb{C}$ is closed.

(iii) If T is selfadjoint, then $\sigma(T) \subseteq \mathbb{R}$.

(iv) If T is symmetric and satisfies $\sigma(T) \subseteq \mathbb{R}$, then T is selfadjoint.

6. The spectral theorem and functional calculus

Theorem 5. *Let $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be selfadjoint. Then there is a unique spectral measure E such that*

$$\langle Tx, y \rangle = \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)x, y \rangle \qquad \text{for all } x \in \operatorname{dom} T,\ y \in \mathcal{H}.$$

If $h : \mathbb{R} \to \mathbb{R}$ is measurable, then

$$\langle h(T)x, y \rangle = \int_{\mathbb{R}} h(\lambda) \, d\langle E(\lambda)x, y \rangle$$

defines a selfadjoint operator $h(T) : \mathcal{H} \supseteq \operatorname{dom} h(T) \to \mathcal{H}$ with domain

$$\operatorname{dom} h(T) := \left\{ x \in \mathcal{H} \;\middle|\; \int_{\mathbb{R}} |h(\lambda)|^2 \, d\langle E(\lambda)x, x \rangle < \infty \right\}.$$