Assignment 1B

Exercise 1:

According to Definition 1.1, we have to show that

(i) D_V is selfadjoint;

(ii) D_V has compact resolvent;

(iii) $\forall a \in \mathcal{A}$: • $\pi(a) \operatorname{dom} D_V \subseteq \operatorname{dom} D_V$ and

• $[D_V, \pi(a)]$ is bounded on $\mathcal{H}$.

(i) This is a general fact:

$T: \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$, $S \in B(\mathcal{H}) \implies (T+S)^* = T^* + S^*$

In particular, if T, S are selfadjoint, then $T+S$ is also selfadjoint.

Proof: Note that

$$\operatorname{dom}(T+S)^* = \{ y \in \mathcal{H} \mid \underset{\in\ \operatorname{dom}(T+S) = \operatorname{dom} T}{x} \mapsto \underbrace{\langle (T+S)x, y\rangle}_{= \langle Tx, y\rangle + \langle Sx, y\rangle} \text{ bounded} \}$$

$$= \{ y \in \mathcal{H} \mid x \mapsto \langle Tx, y \rangle \text{ bounded} \}$$

$$= \operatorname{dom} T^*,$$

since $x \mapsto \langle Sx, y\rangle$ is bounded for every $y \in \mathcal{H}$. Thus:

$$\forall x \in \operatorname{dom} T, \ \forall y \in \operatorname{dom} T^*: \quad \langle (T+S)x, y\rangle = \langle x, (T+S)^* y\rangle$$

$$\langle (T+S)x, y\rangle = \langle Tx, y \rangle + \langle Sx, y\rangle = \langle x, (T^* + S^*)y\rangle$$

$\implies (T+S)^* = T^* + S^*$. $\square$

(ii) Take any $\lambda \in \mathbb{C} \setminus \sigma(\mathcal{D}_V)$ and choose $\lambda_1 \in \mathbb{C} \setminus \sigma(\mathcal{D})$. Put $\lambda_2 := \lambda - \lambda_1 \in \mathbb{C}$. Then

$$\begin{aligned}
&(\mathcal{D}_V - \lambda 1)^{-1} - (\mathcal{D} - \lambda_1 1)^{-1} \\
&\qquad = (\mathcal{D} - \lambda_1 1)^{-1} \underbrace{\big((\mathcal{D} - \lambda_1 1) - (\mathcal{D}_V - \lambda 1) \big)}_{= -(V - \lambda_2 1)} (\mathcal{D}_V - \lambda 1)^{-1}
\end{aligned}$$

$$\Rightarrow (\mathcal{D}_V - \lambda 1)^{-1} = \underbrace{(\mathcal{D} - \lambda_1 1)^{-1}}_{\text{compact}} \underbrace{\Big(1 - (V - \lambda_2 1)(\mathcal{D}_V - \lambda 1)^{-1} \Big)}_{\text{bounded}},$$

which yields that $(\mathcal{D}_V - \lambda 1)^{-1}$ is compact.

(Recall that $K(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$.)

(iii) $\forall a \in \mathcal{A}$:

- $\pi(a)\, \mathrm{dom}\, \mathcal{D}_V \subseteq \mathrm{dom}\, \mathcal{D}_V$ as $\mathrm{dom}\, \mathcal{D}_V = \mathrm{dom}\, \mathcal{D}$
- $[\mathcal{D}_V, \pi(a)] = [\mathcal{D}, \pi(a)] + \underbrace{[V, \pi(a)]}_{\in B(\mathcal{H})}$

extends to a bounded operator on $\mathcal{H}$.

Exercise 2:

(i) Claim: $\partial_j|_{x_0}$, $j = 1, \dots, n$ are $\mathbb{R}$-linearly independent.

Proof: Suppose that we have $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\alpha_1 \partial_1|_{x_0} + \dots + \alpha_n \partial_n|_{x_0} = 0 \quad \text{in } T_{x_0}\mathbb{R}^n.$$

Then, for each $[f]_{x_0} \in C^\infty_{x_0}(\mathbb{R}^n)$,

$$0 = \sum_{j=1}^n \alpha_j \, \partial_j|_{x_0}([f]_{x_0}) = \sum_{j=1}^n \alpha_j \frac{\partial f}{\partial x_j}(x_0).$$

For $i = 1, \dots, n$, we apply this to $[f_i]_{x_0}$ with

$$f_i : \mathbb{R}^n \to \mathbb{R}, \ (x_1, \dots, x_n) \mapsto x_i - x_{0,i}$$

which gives

$$0 = \sum_{j=1}^n \alpha_j \frac{\partial f_i}{\partial x_j}(x_0) = \alpha_i,$$

as desired. $\square$

Claim: $\mathrm{span}_{\mathbb{R}}\{\partial_j|_{x_0} \mid j = 1, \dots, n\} = T_{x_0}\mathbb{R}^n$

Proof: Take any $\delta \in T_{x_0}\mathbb{R}^n$. For every open set $U \subseteq \mathbb{R}^n$ with $x_0 \in U$ and each smooth function $f: U \to \mathbb{R}$, we find by Taylor's theorem on every open ball $B_r(x_0) \subseteq U$ with $r > 0$ a smooth function $\varphi: B_r(x_0) \to \mathbb{R}$ such that

$$f(x) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle + \varphi(x)$$

for all $x \in B_r(x_0)$ and

$$\frac{\varphi(x)}{|x - x_0|} \longrightarrow 0 \quad \text{for } x \to x_0 .$$

In fact, one can write for all $x \in B_r(x_0)$

$$\varphi(x) = \langle g(x), x - x_0 \rangle = \sum_{j=1}^{n} g_j(x) f_i(x)$$

where $g = (g_1, \dots, g_n) : B_r(x_0) \to \mathbb{R}^n$ is given by

$$g_j(x) := \int_0^1 \left[\frac{\partial f}{\partial x_j}(x_0 + s(x - x_0)) - \frac{\partial f}{\partial x_j}(x_0) \right] ds$$

and $f_1, \dots, f_n$ are the functions defined in (i).
Note that $g_1, \dots, g_n$ are smooth with $g_j(x_0) = 0$. Hence

$$\delta([\varphi]_{x_0}) = \delta\left(\sum_{j=1}^{n} [g_j]_{x_0} [f_j]_{x_0} \right)$$

$$= \sum_{j=1}^{n} \left(\delta([g_j]_{x_0}) \underbrace{f_j(x_0)}_{=0} + \underbrace{g_j(x_0)}_{=0} \delta([f_i]_{x_0}) \right)$$

$$= 0,$$

and thus (since $\delta([1]_{x_0}) = 0$)

$$\delta([f]_{x_0}) = \sum_{j=1}^{n} \underbrace{\delta([f_j]_{x_0})}_{=: \alpha_j} \frac{\partial f}{\partial x_j}(x_0)$$

$$= \sum_{j=1}^{n} \alpha_j \, \partial_j|_{x_0}([f]_{x_0})$$

$\square$

(ii) Note that we have an isomorphism

$$\Phi_{i,x_0} : C^\infty_{x_0}(\mathcal{M}) \longrightarrow C^\infty_{\varphi_i(x_0)}(\mathbb{R}^n),$$

$$[f]_{x_0} \longmapsto [f \circ \varphi_i^{-1}]_{\varphi_i(x_0)}$$

and thus an isomorphism

$$\hat{\Phi}_{i,x_0} : T_{\varphi_i(x_0)}\mathbb{R}^n \longrightarrow T_{x_0}\mathcal{M}, \quad \delta \longmapsto \delta \circ \Phi_{i,x_0}.$$

Take the isomorphism from (i), i.e.,

$$\Psi_{\varphi_i(x_0)} : \mathbb{R}^n \longrightarrow T_{\varphi_i(x_0)}\mathbb{R}^n, \quad (v_1, \dots, v_n) \longmapsto \sum_{j=1}^n v_j \, \partial_j|_{\varphi_i(x_0)}.$$

This yields an isomorphism

$$\Theta_{i,x_0} := \hat{\Phi}_{i,x_0} \circ \Psi_{\varphi_i(x_0)} : \mathbb{R}^n \longrightarrow T_{x_0}\mathcal{M},$$

which looks at $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $[f]_{x_0} \in C^\infty_{x_0}(\mathcal{M})$ as

$$\begin{aligned}
\Theta_{i,x_0}(v)([f]_{x_0}) &= \hat{\Phi}_{i,x_0}\Big(\sum_{j=1}^n v_j \, \partial_j|_{\varphi_i(x_0)}\Big)([f]_{x_0}) \\
&= \Big(\sum_{j=1}^n v_j \, \partial_j|_{\varphi_i(x_0)}\Big)\big(\Phi_{i,x_0}([f]_{x_0})\big) \\
&= \Big(\sum_{j=1}^n v_j \, \partial_j|_{\varphi_i(x_0)}\Big)\big([f \circ \varphi_i^{-1}]_{\varphi_i(x_0)}\big) \\
&= \sum_{j=1}^n v_j \big(\partial_j(f \circ \varphi_i^{-1})\big)(\varphi_i(x_0)),
\end{aligned}$$

which is the given expression.

additional question:

What are the transition matrices of the tangent bundle?

$$\begin{tikzcd}
 & (x,\delta)\in\pi^{-1}(U_i\cap U_j)\ni(x,\delta) \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & \\
(x,\Theta_{i,x}^{-1}(\delta))\in(U_i\cap U_j)\times\mathbb{R}^n \arrow[rr] & & (U_i\cap U_j)\times\mathbb{R}^n\ni(x,\Theta_{j,x}^{-1}(\delta))
\end{tikzcd}$$

$$(x,v) \longmapsto \Big(x,\underbrace{\Theta_{j,x}^{-1}(\Theta_{i,x}(v))}_{=S_{ij}(x)v}\Big)$$

We compute by using the chain rule:

$$\begin{aligned}
\Theta_{i,x}(v)([f]_x) &= \sum_{k=1}^{n} v_k\,(\partial_k(f\circ\varphi_i^{-1}))(\varphi_i(x)) \\
&= \sum_{k=1}^{n} v_k\,\big(\underbrace{\partial_k((f\circ\varphi_j^{-1})\circ\psi_{ij}}\big)(\varphi_i(x)) \\
&= \sum_{\ell=1}^{n}(\partial_\ell(f\circ\varphi_j^{-1}))(\underbrace{\psi_{ij}(\varphi_i(x))}_{=\varphi_j(x)})\cdot(\partial_k\psi_{ij}^{\ell})(\varphi_i(x)) \\
&= \sum_{\ell=1}^{n}\Big(\sum_{k=1}^{n} v_k\,(\partial_k\psi_{ij}^{\ell})(\varphi_i(x))\Big)(\partial_\ell(f\circ\varphi_j^{-1}))(\varphi_j(x))
\end{aligned}$$

Thus:

$$\begin{aligned}
\Theta_{j,x}^{-1}(\Theta_{i,x}(v)) &= \Big(\sum_{k=1}^{n} v_k\,(\partial_k\psi_{ij}^{\ell})(\varphi_i(x))\Big)_{\ell=1}^{n} \\
&= \Big((\partial_k\psi_{ij}^{\ell})(\varphi_i(x))\Big)_{\ell,k=1}^{n}\begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix} \\
&= (J\psi_{ij})(\varphi_i(x))\,v\ , \quad \text{i.e. } S_{ij}(x)=(J\psi_{ij})(\varphi_i(x))
\end{aligned}$$