Assignment 2A

Exercise 1:

(i) We define $$E^* := \coprod_{x\in X} E_x^* = \{(x,f) \mid x\in X, f\in E_x^*\}$$

and $\pi^* : E^* \to X,\ (x,f) \mapsto x$.

Consider a (maximal) bundle atlas of the bundle $\pi : E \to X$, say $\mathcal{A} = \{(U_i, \tau_i) \mid i\in I\}$ with local trivialization $\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^n$.

By definition, $\tau_i|_{E_x} : E_x \to \{x\}\times \mathbb{K}^n = \mathbb{K}^n$ is an isomorphism for each $x\in U_i$; thus

$$(\tau_i|_{E_x})' : (\mathbb{K}^n)^* \to E_x^*,\ g \mapsto g\circ(\tau_i|_{E_x})$$

(the dual operator) is an isomorphism as well.

Fix an isomorphism $\Phi : \mathbb{K}^n \to (\mathbb{K}^n)^*$. Define

$$\tau_i^* : (\pi^*)^{-1}(U_i) \to U_i \times \mathbb{K}^n$$

fibre-wise by

$$\tau_i^*|_{E_x^*} := \left((\tau_i|_{E_x})' \circ \Phi\right)^{-1} : E_x^* \to \mathbb{K}^n = \{x\}\times\mathbb{K}^n.$$

We endow E^* with the topology defined by

$W \subseteq E^*$ open $:\Leftrightarrow \forall i\in I : \tau_i^*(U_i\cap W) \subseteq U_i \times \mathbb{K}^n$ open

Then E^* is a vector bundle with bundle atlas

$$\mathcal{A}^* = \{ (U_i, \tau_i^*) \mid i \in I \}.$$

(ii) The transition maps σ_{ij}^* of E^* are of the form

$$\sigma_{ij}^*(x,v) = (x, S_{ij}^*(x)v) \quad \forall (x,v) \in (U_i \cap U_j) \times \mathbb{K}^n$$

with the transition matrices

$$S_{ij}^* : U_i \cap U_j \to GL_n(\mathbb{K})$$

that are determined at any point $x \in U_i \cap U_j$ by

$$L_{\mathcal{E}}^{\mathcal{E}}(S_{ij}^*(x)) = \tau_j^*|_{E_x^*} \circ (\tau_i^*|_{E_x^*})^{-1}$$

$\boxed{\mathcal{E} = \{e_1, \dots, e_n\} \text{ standard basis of } \mathbb{K}^n; \; \mathcal{E}^* = \{e_1^*, \dots, e_n^*\} \text{ dual basis to } \mathcal{E}; \text{ suppose } \Phi(e_i) = e_i^* \;\; \forall i = 1, \dots, n}$

$$= \left((\tau_j|_{E_x})' \circ \Phi \right)^{-1} \circ \left((\tau_i|_{E_x})' \circ \Phi \right)$$

$$= \Phi^{-1} \circ \Big(\underbrace{\tau_i|_{E_x} \circ (\tau_j|_{E_x})^{-1}}_{= L_{\mathcal{E}}^{\mathcal{E}}(S_{ji}(x))} \Big)' \circ \underset{L_{\mathcal{E}^*}^{\mathcal{E}}(1_n)}{\underset{\|}{\Phi}}$$

$$= \Phi^{-1} \circ L_{\mathcal{E}^*}^{\mathcal{E}^*}(S_{ji}(x)^T) \circ \Phi = L_{\mathcal{E}}^{\mathcal{E}}(S_{ji}(x)^T)$$

where $S_{ij} : U_i \cap U_j \to GL_n(\mathbb{K})$ are the transition matrices for E; hence $S_{ij}^*(x) = S_{ji}(x)^T = (S_{ij}(x)^{-1})^T$.

In particular, if E is smooth, then E^* is smooth as well.

Reminder:

- V, W $\mathbb{K}$-vector spaces with bases $\mathcal{V}, \mathcal{W}$, say

$$\mathcal{V} = \{v_1, \dots, v_n\} \quad \text{and} \quad \mathcal{W} = \{w_1, \dots, w_m\}$$

- $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in \mathbb{K}^{m \times n} \rightsquigarrow L^{\mathcal{V}}_{\mathcal{W}}(A) : V \to W$ linear map

$$v_j \mapsto \sum_{i=1}^{m} a_{ij} w_i$$

- $\mathcal{V}^* = \{v_1^*, \dots, v_n^*\}$ and $\mathcal{W}^* = \{w_1^*, \dots, w_m^*\}$

⌢ dual bases of V^* and W^*, then

$$L^{\mathcal{V}}_{\mathcal{W}}(A)' = L^{\mathcal{W}^*}_{\mathcal{V}^*}(A^T) \qquad \forall A \in \mathbb{K}^{m \times n}$$

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