Assignment 2B

Exercise 1:

(i) Take $D \in \operatorname{der} C^\infty(V)$. We have to show that

$$\Psi(D): V \to T\mathcal{M}, \quad x \mapsto (\Psi(D))(x)$$

is a smooth section in the sense of Def 2.7.

- Since $(\Psi(D))(x) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$, $\pi \circ \Psi(D) = id_V$ is clear.

- Take a local trivialization $\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ of the tangent bundle $\pi : T\mathcal{M} \to \mathcal{M}$; recall that

$$\tau_i(\delta) = (x, \Theta_{i,x}^{-1}(\delta)) \quad \text{for all } x \in U.$$

Then, for every $x \in U \cap V$, we see that

$$\tau_i((\Psi(D))(x)) = (x, \Theta_{i,x}^{-1}((\Psi(D))(x))) = (x, v(x))$$

where $v(x) = (v_1(x), \dots, v_n(x)) \in \mathbb{R}^n$ is such that

$$(\Psi(D))(x)([f]_x) = \sum_{j=1}^n v_j(x)\, \partial_j (f \circ \varphi_i^{-1})(\varphi_i(x)) \;\; \text{“}\!\!= \Theta_{i,x}(v(x))([f]_x)$$

for every $[f]_x \in C_x^\infty(\mathcal{M})$.

Recall that each $\delta \in T_{\varphi_i(x)}\mathbb{R}^n$ can be written as

$$\delta = \sum_{j=1}^n \delta([g_j]_{\varphi_i(x)})\, \partial_j|_{\varphi_i(x)}$$

where $g_j : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto x_j$. Since

$$\hat{\Phi}_{i,x} : T_{\varphi_i(x)} \mathbb{R}^n \to T_x \mathcal{M}, \quad \delta \mapsto \delta \circ \Phi_{i,x}$$

with
$$\Phi_{i,x} : C_x^\infty(\mathcal{M}) \to C^\infty_{\varphi_i(x)}(\mathbb{R}^n),$$
$$[f]_x \mapsto [f \circ \varphi_i^{-1}]_{\varphi_i(x)}$$

is an isomorphism, we get that

$$\delta := \hat{\Phi}_{i,x}^{-1}\Big((\Psi(D))(x) \Big) = \sum_{j=1}^n \delta([g_j]_{\varphi_i(x)})\, \partial_j|_{\varphi_i(x)}$$

$$\Rightarrow (\Psi(D))(x) = \sum_{j=1}^n \delta([g_j]_{\varphi_i(x)})\, \Phi_{i,x}(\partial_j|_{\varphi_i(x)})$$

$$\Rightarrow (\Psi(D))(x)([f]_x) = \sum_{j=1}^n v_j(x)\, \partial_j(f \circ \varphi_i^{-1})(\varphi_i(x))$$

for all $[f]_x \in C_x^\infty(\mathcal{M})$, where

$$\begin{aligned}
v_j(x) &:= \delta([g_j]_{\varphi_i(x)}) \\
&= \hat{\Phi}_{i,x}^{-1}\Big((\Psi(D))(x) \Big)([g_j]_{\varphi_i(x)}) \\
&= (\Psi(D))(x)\Big(\Phi_{i,x}^{-1}([g_j]_{\varphi_i(x)}) \Big) \\
&= (\Psi(D))(x)\Big([g_j \circ \varphi_i]_x \Big) \\
&= D|_x\Big(\rho \cdot (g_j \circ \varphi_i)|_V \Big) \\
&= D\Big(\rho \cdot (g_j \circ \varphi_i)|_V \Big)(x)
\end{aligned}$$

Hence $v_j = D(\rho \cdot (g_j \circ \varphi_i)|_V)|_{U \cap V} \in C^\infty(U \cap V)$.

This shows that $\Psi(D) \in \mathfrak{X}(V)$.

(ii) ① Take $D \in \operatorname{der} C^\infty(V)$ and put $X := \Psi(D) \in \mathfrak{X}(V)$.

We want to compute $\Phi(X)$. For every $f \in C^\infty(V)$ and $x \in V$, we have that

$$
\begin{aligned}
(\Phi(X) f)(x) &= X(x)([f]_x) \\
&= \Psi(D)(x)([f]_x) \\
&= D|_x(\rho \cdot f|_V) \\
&= D(\rho|_V \cdot f)(x) \\
&= \underbrace{D(\rho|_V)(x)}_{= D(1|_V)(x) = 0} \cdot f(x) + \underbrace{(\rho|_V)(x)}_{=1} D(f)(x) \\
&= (D(f))(x),
\end{aligned}
$$

i.e., $\Phi(X) = D$; hence $\Phi \circ \Psi = \mathrm{id}_{\operatorname{der} C^\infty(V)}$

② Take $X \in \mathfrak{X}(V)$ and put $D := \Phi(X) \in \operatorname{der} C^\infty(V)$.

We want to compute $\Psi(D)$. For every $x \in V$ and $[f]_x \in C_x^\infty(M)$, we have that

$$
\begin{aligned}
(\Psi(D))(x)([f]_x) &= D|_x(\rho \cdot f|_V) \\
&= D(\rho \cdot f|_V)(x)
\end{aligned}
$$

$$= \big(\Phi(X)(\rho \cdot f|_V)\big)(x)$$

$$= X(x)\big(\underbrace{[\rho \cdot f|_V]_x}_{= [f]_x}\big)$$

$$= X(x)\big([f]_x\big),$$

i.e., $\Psi(D) = X$; hence $\Psi \circ \Phi = id_{\mathfrak{X}(V)}$.