Assignment 3A

Exercise 1:

Suppose that $\operatorname{supp}(f) \subseteq U_1 \cap U_2$ for two local charts (U_1, φ_1) and (U_2, φ_2) in $\mathcal{A}$.

Consider the matrix $G^i := (g^i_{k,l})^n_{k,l=1}$ of functions $g^i_{k,l} \in C^\infty(\varphi_i(U_i))$ that are defined by

$$g^i_{k,l}(\varphi_i(x)) = g_x\left((d\varphi_i)(x)^{-1}(\partial_k|_{\varphi_i(x)}), (d\varphi_i)(x)^{-1}(\partial_l|_{\varphi_i(x)})\right)$$

for every $x \in U_i$, $i = 1, 2$. Consider the transition map

$$\psi : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2), \quad \psi := \varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}.$$

We have to prove that (note that $\operatorname{supp} f \subseteq U_1 \cap U_2$):

$$\underbrace{\int_{\varphi_1(U_1 \cap U_2)} f \circ \varphi_1^{-1} \sqrt{\det G^1}\, d\lambda^n}_{=: I_1} = \underbrace{\int_{\varphi_2(U_1 \cap U_2)} f \circ \varphi_2^{-1} \sqrt{\det G^2}\, d\lambda^n}_{=: I_2}$$

By the transformation formula,

$$I_2 = \int_{\varphi_1(U_1 \cap U_2)} \underbrace{f \circ \varphi_2^{-1} \circ \psi}_{= f \circ \varphi_1^{-1}} \sqrt{\det(G^2 \circ \psi)}\, |\det \underbrace{J\psi}_{=(\partial_l \psi_k)^n_{k,l=1}}|\, d\lambda^n.$$

Thus, it suffices to prove that

$$(J\psi)^T \cdot (G^2 \circ \psi) \cdot J\psi = G^1 \quad \text{on } \varphi_1(U_1 \cap U_2).$$

For each $x \in U_1 \cap U_2$, we must see that

$$(\partial\psi)(\varphi_1(x))^T \underbrace{G^2(\psi(\varphi_1(x)))}_{= G^2(\varphi_2(x))} (\partial\psi)(\varphi_1(x)) = G^1(\varphi_1(x));$$

it suffices to prove that for each $k = 1, \dots, n$

$$\sum_{p=1}^{n} (\partial_k \psi_p)(\varphi_1(x))\, (d\varphi_2)(x)^{-1}\left(\partial_p|_{\varphi_2(x)}\right) = (d\varphi_1)(x)^{-1}\left(\partial_k|_{\varphi_1(x)}\right)$$

$$= (d\varphi_2)(x)^{-1}(\delta_k) \text{ with } \delta_k := \sum_{p=1}^{n} (\partial_k \psi_p)(\varphi_1(x))\, \partial_p|_{\varphi_2(x)}$$

$$\iff \quad \delta_k = \underbrace{(d\varphi_2)(x)(d\varphi_1)(x)^{-1}}_{\overset{(*)}{=} (d\psi)(\varphi_1(x))}\left(\partial_k|_{\varphi_1(x)}\right) \qquad (**)$$

for $(*)$: $\left((d\psi)(\varphi_1(x))\underbrace{(d\varphi_1)(x)\,\delta}\right)\left([f]_{\varphi_2(x)}\right)$, $\varphi_2(x) = \psi(\varphi_1(x))$

$$= \left((d\varphi_1)(x)\,\delta\right)\left([f\circ\psi]_{\varphi_1(x)}\right)$$

$$= \delta\left([f\circ\underbrace{\psi\circ\varphi_1}_{=\varphi_2}]_x\right)$$

$$= \delta\left([f\circ\varphi_2]_x\right) = (d\varphi_2)(x)\left([f]_{\varphi_2(x)}\right)$$

for $(**)$: $\left((d\psi)(\varphi_1(x))\left(\partial_k|_{\varphi_1(x)}\right)\right)\left([f]_{\varphi_2(x)}\right)$

$$= \partial_k|_{\varphi_1(x)}\left([f\circ\psi]_{\varphi_1(x)}\right)$$

$$= \partial_k(f\circ\psi)(\varphi_1(x))$$

$$= \sum_{p=1}^{n} (\partial_p f)(\varphi_2(x)) \cdot (\partial_k \psi_p)(\varphi_1(x)) = \delta_k\left([f]_{\varphi_2(x)}\right)$$

Exercise 2:

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Hilbert space and let $(V_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ be its complexification.

Claim: $$\langle \bar{v} \wedge \eta, \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}} = \langle \eta, v \llcorner \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}}$$

for all $v \in V_{\mathbb{C}}$ and $\omega, \eta \in \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}$

Proof: Let $\eta \in \Lambda^p_{\mathbb{C}} V_{\mathbb{C}}$ and $\omega \in \Lambda^q_{\mathbb{C}} V_{\mathbb{C}}$. Then

$$\langle \bar{v} \wedge \eta, \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}} = 0 = \langle \eta, v \llcorner \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}}$$

whenever $q \neq p+1$. Thus, it suffices to consider

$$\eta = \eta_1 \wedge \dots \wedge \eta_p \in \Lambda^p_{\mathbb{C}} V_{\mathbb{C}} \quad \text{and}$$

$$\omega = \omega_0 \wedge \dots \wedge \omega_p \in \Lambda^{p+1}_{\mathbb{C}} V_{\mathbb{C}} .$$

Then:

$$\langle \bar{v} \wedge \eta, \omega \rangle_{\Lambda^{p+1}_{\mathbb{C}} V_{\mathbb{C}}} = \det \underbrace{\begin{pmatrix} \langle \bar{v}, \omega_0 \rangle_{\mathbb{C}} & \langle \bar{v}, \omega_1 \rangle_{\mathbb{C}} & \cdots & \langle \bar{v}, \omega_p \rangle_{\mathbb{C}} \\ \hline \langle \eta_1, \omega_0 \rangle_{\mathbb{C}} & \langle \eta_1, \omega_1 \rangle_{\mathbb{C}} & \cdots & \langle \eta_1, \omega_p \rangle_{\mathbb{C}} \\ \vdots & & & \vdots \\ \langle \eta_p, \omega_0 \rangle_{\mathbb{C}} & \langle \eta_p, \omega_1 \rangle_{\mathbb{C}} & \cdots & \langle \eta_p, \omega_p \rangle_{\mathbb{C}} \end{pmatrix}}_{=:M}$$

Laplace for first row, $i=1$

$$= \sum_{j=1}^{p+1} (-1)^{1+j} \langle \bar{v}, \omega_{j-1} \rangle_{\mathbb{C}} \det(M_{1,j}), \quad \text{where}$$

$$M_{1,j} = \begin{pmatrix} \langle \eta_1, \omega_0 \rangle_{\mathbb{C}} & \cdots & \langle \eta_1, \omega_{j-1} \rangle_{\mathbb{C}} & \cdots & \langle \eta_1, \omega_p \rangle_{\mathbb{C}} \\ \vdots & & \vdots & & \vdots \\ \langle \eta_p, \omega_0 \rangle_{\mathbb{C}} & \cdots & \langle \eta_p, \omega_{j-1} \rangle_{\mathbb{C}} & \cdots & \langle \eta_p, \omega_p \rangle_{\mathbb{C}} \end{pmatrix}$$

(jth column removed)

and hence

$$\det(M_{1,j}) = \langle \eta_1 \wedge \dots \wedge \eta_p, \omega_0 \wedge \dots \wedge \widehat{\omega_{j-1}} \wedge \dots \wedge \omega_p \rangle_{\Lambda^\bullet_{\mathbb{C}} V_{\mathbb{C}}}$$

Thus:

$$\langle \bar{v} \wedge \eta, \omega \rangle_{\Lambda^\bullet_{\mathbb{C}} V_{\mathbb{C}}} = \langle \eta_1 \wedge \ \wedge \eta_p, \bar{v} \llcorner (\omega_0 \wedge \dots \wedge \omega_p) \rangle_{\Lambda^\bullet_{\mathbb{C}} V_{\mathbb{C}}}$$

because

$$\bar{v} \llcorner (\omega_0 \wedge \dots \wedge \omega_p) = \sum_{j=1}^{p+1} (-1)^{1+j} \langle \omega_{j-1}, \bar{v} \rangle_{\mathbb{C}} \, \omega_0 \wedge \dots \wedge \widehat{\omega_{j-1}} \wedge \dots \wedge \omega_p$$

$\square$

Apply this to $V = T^*_x \mathcal{M}$ and $v = (df)(x)$ for $x \in \mathcal{M}$; note $\bar{v} = (d\bar{f})(x)$. This gives

$$\langle (d\bar{f})(x) \wedge \eta(x), \omega(x) \rangle_{\Lambda^\bullet_{\mathbb{C}} T^*_x \mathcal{M}_{\mathbb{C}}} = \langle \eta(x), (df)(x) \llcorner \omega(x) \rangle_{\Lambda^\bullet_{\mathbb{C}} T^*_x \mathcal{M}_{\mathbb{C}}}$$

and after integration over $\mathcal{M}$

$$\langle d\bar{f} \wedge \eta, \omega \rangle_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})} = \langle \eta, df \llcorner \omega \rangle_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})},$$

as desired