UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Tobias Mai

Assignments for the lecture
Introduction to Noncommutative Differential Geometry
Summer term 2019

Assignment 3B
for the tutorial on *Tuesday, May 21, 10:15 am* (in Seminar Room 10)

Exercise 1. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Hilbert space and let $(V_\mathbb{C}, \langle\cdot,\cdot\rangle_\mathbb{C})$ its complexification. On the finite dimensional complex Hilbert space $\mathcal{F}_-(V_\mathbb{C}) := \bigoplus_{p\geq 0} \bigwedge^p_\mathbb{C} V_\mathbb{C}$, called the *antisymmetric Fock space*, we introduce for each $v \in V_\mathbb{C}$ the linear operators

$$a(v): \ \mathcal{F}_-(V_\mathbb{C}) \to \mathcal{F}_-(V_\mathbb{C}), \quad w \mapsto \overline{v} \llcorner w$$

and

$$a^*(v): \ \mathcal{F}_-(V_\mathbb{C}) \to \mathcal{F}_-(V_\mathbb{C}), \quad w \mapsto v \wedge w.$$

Prove the following assertions:

(i) For each $v \in V_\mathbb{C}$, we have that $a(v)^* = a^*(v)$.

(ii) For $v_1, v_2 \in V_\mathbb{C}$, we have that

$$a(v_2)a^*(v_1) + a^*(v_1)a(v_2) = \langle v_1, v_2\rangle_\mathbb{C} \operatorname{id}_{\mathcal{F}_-(V_\mathbb{C})}.$$

(iii) For each $v \in V_\mathbb{C}$, the operator

$$x(v) := i(a^*(v) - a(\overline{v}))$$

satisfies $x(v)^* = x(\overline{v})$. If $v \in V_\mathbb{C}$ is real (i.e., if $\overline{v} = v$ holds), the operator $x(v)$ is selfadjoint and has the norm $\|x(v)\| = \|v\|$.
How does $x(v)$ for $v \in V_\mathbb{C}$ relate to the Clifford multiplication from the left with v?

Hint: For the statement about the operator norm, compute $x(v)^2$ for real $v \in V_\mathbb{C}$.