Exercise 1. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real Hilbert space and let $(V_C, \langle \cdot, \cdot \rangle_C)$ its complexification. On the finite dimensional complex Hilbert space $\mathcal{F}_-(V_C) := \bigoplus_{p \geq 0} \Lambda^p V_C$, called the antisymmetric Fock space, we introduce for each $v \in V_C$ the linear operators

$$a(v) : \mathcal{F}_-(V_C) \to \mathcal{F}_-(V_C), \quad w \mapsto \overline{v} \wedge w$$

and

$$a^*(v) : \mathcal{F}_-(V_C) \to \mathcal{F}_-(V_C), \quad w \mapsto v \wedge w.$$ 

Prove the following assertions:

(i) For each $v \in V_C$, we have that $a(v)^* = a^*(v)$.

(ii) For $v_1, v_2 \in V_C$, we have that

$$a(v_2)a^*(v_1) + a^*(v_1)a(v_2) = \langle v_1, v_2 \rangle_C \id_{\mathcal{F}_-(V_C)}.$$ 

(iii) For each $v \in V_C$, the operator

$$x(v) := i(a^*(v) - a(v))$$

satisfies $x(v)^* = x(\overline{v})$. If $v \in V_C$ is real (i.e., if $\overline{v} = v$ holds), the operator $x(v)$ is selfadjoint and has the norm $\|x(v)\| = \|v\|$.

How does $x(v)$ for $v \in V_C$ relate to the Clifford multiplication from the left with $v$?

**Hint:** For the statement about the operator norm, compute $x(v)^2$ for real $v \in V_C$. 