Assignment 3B

Exercise 1:

(i) In Exercise 2 on Sheet 3A, we have already shown that for all $\upsilon \in \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}} = \mathcal{F}_{-}(V_{\mathbb{C}})$ and $\upsilon \in V_{\mathbb{C}}$

$$\langle \upsilon \wedge \eta, \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}} = \langle \eta, \bar{\upsilon} \llcorner \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}}.$$

i.e., $\langle a^*(\upsilon)\eta, \omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}} = \langle \eta, a(\upsilon)\omega \rangle_{\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}}$.

$$\Longrightarrow \quad a(\upsilon)^* = a^*(\upsilon).$$

(ii) Take any $\omega = \omega_1 \wedge \ldots \wedge \omega_p \in \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}$. Then

$$
\begin{aligned}
a(\upsilon_2) a^*(\upsilon_1)\omega &= a(\upsilon_2)(\upsilon_1 \wedge \omega) \\
&= \bar{\upsilon}_2 \llcorner (\upsilon_1 \wedge \omega_1 \wedge \ldots \wedge \omega_p) \\
&= \langle \upsilon_1, \upsilon_2 \rangle_{\mathbb{C}}\, \omega - \sum_{k=1}^{p} (-1)^{k+1} \langle \omega_k, \upsilon_2 \rangle_{\mathbb{C}}\, \upsilon_1 \wedge \omega_1 \wedge \ldots \wedge \hat{\omega}_k \wedge \ldots \wedge \omega_p \\
&= \langle \upsilon_1, \upsilon_2 \rangle_{\mathbb{C}}\, \omega - \upsilon_1 \wedge \left(\sum_{k=1}^{p} (-1)^{k+1} \langle \omega_k, \upsilon_2 \rangle_{\mathbb{C}}\, \omega_1 \wedge \ldots \wedge \hat{\omega}_k \wedge \ldots \wedge \omega_p \right) \\
&= \langle \upsilon_1, \upsilon_2 \rangle_{\mathbb{C}}\, \omega - \upsilon_1 \wedge (\bar{\upsilon}_2 \llcorner \omega) \\
&= \langle \upsilon_1, \upsilon_2 \rangle_{\mathbb{C}}\, \omega - a^*(\upsilon_1) a(\upsilon_2)\omega
\end{aligned}
$$

$$\Longrightarrow \quad a(\upsilon_2)a^*(\upsilon_1) + a^*(\upsilon_1)a(\upsilon_2) = \langle \upsilon_1, \upsilon_2 \rangle\, id_{\mathcal{F}_{-}(V_{\mathbb{C}})}.$$

(iii) • For $v \in V_{\mathbb{C}}$, we have by (i)

$$X(v)^* = -i\left(a^*(v)^* - a(\bar{v})^*\right)$$

$$= -i\left(a(v) - a^*(\bar{v})\right)$$

$$= i\left(a^*(\bar{v}) - a(v)\right) = X(\bar{v}).$$

Thus, if $v = \bar{v}$, then $X(v)^* = X(v)$.

• Take $v_1, v_2 \in V_{\mathbb{C}}$ which are real. Then

$$X(v_1)X(v_2) + X(v_2)X(v_1) = 2\langle v_1, v_2\rangle \, id_{\mathcal{F}(V_{\mathbb{C}})}.$$

Indeed:

$$X(v_1)X(v_2) = -\left(a^*(v_1) - a(v_1)\right)\left(a^*(v_2) - a(v_2)\right)$$

$$= -\Big(\underbrace{a^*(v_1)a^*(v_2)}_{①} - \underbrace{a(v_1)a^*(v_2)}_{②} - \underbrace{a^*(v_1)a(v_2)}_{③} + \underbrace{a(v_1)a(v_2)}_{④}\Big)$$

$$X(v_2)X(v_1) = -\left(a^*(v_2) - a(v_2)\right)\left(a^*(v_1) - a(v_1)\right)$$

$$= -\Big(\underbrace{a^*(v_2)a^*(v_1)}_{①} - \underbrace{a(v_2)a^*(v_1)}_{③} - \underbrace{a^*(v_2)a(v_1)}_{②} + \underbrace{a(v_2)a(v_1)}_{④}\Big)$$

①: $a^*(v_1)a^*(v_2) = -a^*(v_2)a^*(v_1)$

④: applying $*$ to ① gives $a(v_1)a(v_2) = -a(v_2)a(v_1)$

② and ③: each gives $\langle v_1, v_2\rangle \left(= \langle \bar{v}_2, \bar{v}_1\rangle = \langle v_2, v_1\rangle\right)$

- Thus, for each real $v \in V_{\mathbb{C}}$:

$$x(v)^2 = \|v\|^2 \, id_{\mathcal{F}_-(V_{\mathbb{C}})}$$

$$\Rightarrow \quad \|x(v)\|^2 = \|x(v)^* x(v)\|$$

$$= \|x(v)^2\| \quad \text{since } x(v)^* = x(v)$$

$$= \|v\|^2$$

$$\Rightarrow \quad \|x(v)\| = \|v\|.$$

- If $v \in V_{\mathbb{C}}$ is real, then

$$x(v)\,\omega = i\,(v \wedge \omega - v \,\llcorner\, \omega)$$

$$= i \; v \cdot \omega.$$