UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Introduction to Noncommutative Differential Geometry
Summer term 2019

Assignment 4 A & B
for the tutorial on *Tuesday, June 4, 10:15 am* (in Seminar Room 10)

Note that there will be ***no lecture*** *on Monday, May 27; the* ***next lecture*** *is accordingly on Monday, June 3. Thus, there is only this (slightly extended) exercise sheet for the* ***next problem session*** *on Tuesday, June 4.*

Exercise 1. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with the faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$. Consider the state space $S(A)$ for the associated C^*-algebra $A := \overline{\pi(\mathcal{A})}^{\|\cdot\|} \subseteq B(\mathcal{H})$. Prove the following assertions:

(i) If the image of $\{a \in \mathcal{A} \mid \|[\mathcal{D}, \pi(a)]\| \leq 1\}$ in the quotient Banach space $A/\mathbb{C}1$ is a norm bounded set, then the spectral distance satisfies $d_\mathcal{D}(\varphi, \psi) < \infty$ for all $\varphi, \psi \in S(A)$ and induces a metric $d_\mathcal{D} : S(A) \times S(A) \to [0, \infty)$.

(ii) For all $\varphi, \psi \in S(A)$, we have that

$$d_\mathcal{D}(\varphi, \psi) = \sup\big\{|\psi(\pi(a)) - \varphi(\pi(a))| \bigm| a = a^* \in \mathcal{A} : \ \|[\mathcal{D}, \pi(a)]\| \leq 1\big\}.$$

Hint: In order to prove "$\leq$", establish first that the set $\{a \in \mathcal{A} \mid \|[\mathcal{D}, \pi(a)]\| \leq 1\}$ is closed under the following maps: $a \mapsto \zeta a$ for each $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $a \mapsto a^*$, $a \mapsto \mathrm{Re}(a) = \frac{1}{2}(a + a^*)$, and $a \mapsto \mathrm{Im}(a) = \frac{1}{2i}(a - a^*)$.

Exercise 2. Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples with the faithful $*$-representations $\pi_1 : \mathcal{A}_1 \to B(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A}_2 \to B(\mathcal{H}_2)$, respectively. We call these two spectral triples *equivalent*, if there exists a $*$-isomorphism $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ and a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(a)U^* = \pi_2(\Phi(a))$ for all $a \in \mathcal{A}_1$ and $U\mathcal{D}_1U^* = \mathcal{D}_2$. Show that in this case $\mathrm{ad}_U : B(\mathcal{H}_1) \to B(\mathcal{H}_2), x \mapsto UxU^*$ is an isometry which satisfies $\mathrm{ad}_U(A_1) = A_2$, where A_1 and A_2 are the C^*-algebras associated to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and prove that $\mathrm{ad}_U^* : S(A_2) \to S(A_1), \varphi \mapsto \varphi \circ \mathrm{ad}_U$ defines an isometry for the spectral distances, i.e.,

$$d_{\mathcal{D}_1}(\mathrm{ad}_U^* \varphi, \mathrm{ad}_U^* \psi) = d_{\mathcal{D}_2}(\varphi, \psi) \qquad \text{for all } \varphi, \psi \in S(A_2).$$

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Exercise 3. Consider the complex unital $*$-algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ with entry-wise operations. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional complex Hilbert spaces and put $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Define the $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ by

$$\pi(a) := \begin{pmatrix} a_1 \, \mathrm{id}_{\mathcal{H}_1} & 0 \\ 0 & a_2 \, \mathrm{id}_{\mathcal{H}_2} \end{pmatrix} \qquad \text{for all } a = (a_1, a_2) \in \mathcal{A}.$$

Further, take any linear operator $M : \mathcal{H}_1 \to \mathcal{H}_2$ and consider the operator

$$\mathcal{D} := \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

(i) Verify that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple. Compute for each $a = (a_1, a_2) \in \mathcal{A}$ the commutator $[\mathcal{D}, \pi(a)]$ and show that its norm is given by $\|[\mathcal{D}, \pi(a)]\| = |a_2 - a_1| \|M\|$.

(ii) Consider the states $\delta_1, \delta_2 : \mathcal{A} \to \mathbb{C}$ that are respectively given by $\delta_1(a) = a_1$ and $\delta_2(a) = a_2$ for each $a = (a_1, a_2) \in \mathcal{A}$. Compute the spectral distance $d_{\mathcal{D}}(\delta_1, \delta_2)$.

(iii) Show that the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *even*, i.e., there is a selfadjoint operator $\Gamma \in B(\mathcal{H})$ with the properties that $\Gamma^2 = \mathrm{id}_{\mathcal{H}}$, $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$, and $\pi(a)\Gamma = \Gamma\pi(a)$ for all $a \in \mathcal{A}$. We call Γ a *grading* on $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.