Assignment 4 A & B

Exercise 1

We put $\mathcal{B} := \{ a \in \mathcal{A} \mid \| [D, \pi(a)] \| \leq 1 \}$ and $B := \pi(\mathcal{B})$.

(i) (1) Consider the quotient map

$$q : A \to A/\mathbb{C}1, \quad a \mapsto [a].$$

Clearly, for $\varphi, \psi \in S(A)$

$$d_D(\varphi, \psi) = \sup \{ |\psi(\pi(a)) - \varphi(\pi(a))| \mid a \in \mathcal{B} \}$$

$$= \sup \{ |\psi(a) - \varphi(a)| \mid a \in B \}$$

$$= \sup \{ |f_{\varphi,\psi}(a)| \mid a \in B \},$$

where $f_{\varphi,\psi} : A \to \mathbb{C}$, $a \mapsto \psi(a) - \varphi(a)$. Note that $f_{\varphi,\psi}$ induces a continuous linear functional $\tilde{f}_{\varphi,\psi}$ by

$$\begin{tikzcd} A \arrow[dr, "f_{\varphi,\psi}"] \arrow[d, "q"'] & \\ A/\mathbb{C}1 \arrow[r, dashed, "\exists!\, \tilde{f}_{\varphi,\psi}"'] & \mathbb{C} \end{tikzcd}$$

Thus: $d_D(\varphi, \psi) = \sup \{ |\tilde{f}_{\varphi,\psi}([a])| \mid [a] \in q(B) \}$

$$\Rightarrow d_D(\varphi, \psi) \leq \| \tilde{f}_{\varphi,\psi} \| \cdot \sup_{[a] \in q(B)} \| [a] \| < \infty.$$

② That $d_{\mathcal{D}}$ is symmetric and satisfies the triangle inequality is obvious

<u>Claim</u>: $\varphi, \psi \in S(A)$: $d_{\mathcal{D}}(\varphi,\psi) = 0 \Rightarrow \varphi = \psi$.

<u>Proof</u>: $d_{\mathcal{D}}(\varphi,\psi) = 0$ implies that $\varphi|_{\mathcal{B}} = \psi|_{\mathcal{B}}$;

since span $\mathcal{B} = \pi(\mathcal{A})$, we get $\varphi|_{\pi(\mathcal{A})} = \psi|_{\pi(\mathcal{A})}$;

finally, since $\overline{\pi(\mathcal{A})}^{\|\cdot\|} = A$, we conclude $\varphi = \psi$. □

Thus, we have a metric $d_{\mathcal{D}} : S(A) \times S(A) \to [0,\infty)$.

(ii) <u>Claim</u>: $\mathcal{B}$ is closed under the following maps

(a) $a \mapsto \varsigma a$ for $\varsigma \in \mathbb{C}$ with $|\varsigma| = 1$;

(b) $a \mapsto a^*$

(c) $a \mapsto \mathrm{Re}(a)$ and $a \mapsto \mathrm{Im}(a)$

<u>Proof</u>: (a) $\varsigma a \in \mathcal{A}$ and $[\mathcal{D}, \pi(\varsigma a)] = \varsigma [\mathcal{D}, \pi(a)]$;

(b) $a^* \in \mathcal{A}$ and $[\mathcal{D}, \pi(a^*)] = -[\mathcal{D}, \pi(a)]^*$;

(c)
$$
\begin{aligned}
[\mathcal{D}, \pi(\mathrm{Re}(a))] &= \tfrac{1}{2}\big([\mathcal{D}, \pi(a)] + [\mathcal{D}, \pi(a^*)]\big) \\
&\overset{(b)}{=} \tfrac{1}{2}\big([\mathcal{D}, \pi(a)] - [\mathcal{D}, \pi(a)]^*\big) \\
&= i \cdot \mathrm{Im}([\mathcal{D}, \pi(a)])
\end{aligned}
$$

and analogously

$[\mathcal{D}, \pi(\mathrm{Im}(a))] = -i\, \mathrm{Re}([\mathcal{D}, \pi(a)])$. □

Define

$$d_{\mathcal{D}}^{sa}(\varphi,\psi) := \sup\{|\psi(\pi(a)) - \varphi(\pi(a))| \mid a = a^* \in \mathcal{A} : \|[\mathcal{D},\pi(a)]\| \leq 1\}.$$

<u>Claim</u> : $d_{\mathcal{D}}^{sa}(\varphi,\psi) = d_{\mathcal{D}}(\varphi,\psi)$

<u>Proof</u>: $d_{\mathcal{D}}^{sa}(\varphi,\psi) \leq d_{\mathcal{D}}(\varphi,\psi)$ is obvious; to prove $d_{\mathcal{D}}^{sa}(\varphi,\psi) \geq d_{\mathcal{D}}(\varphi,\psi)$, we proceed as follows:

Take any $\varepsilon > 0$. We find $a \in \mathcal{B}$ such that

$$|\psi(\pi(a)) - \varphi(\pi(a))| \geq d_{\mathcal{D}}(\varphi,\psi) - \varepsilon. \tag{1}$$

Now, choose $\varsigma \in \mathbb{C}$ with $|\varsigma| = 1$ such that

$$\varsigma(\psi(\pi(a)) - \varphi(\pi(a))) \in \mathbb{R} \tag{2}$$

and put $\tilde{a} := \mathrm{Re}(\varsigma a)$. Note that $\tilde{a} = \tilde{a}^* \in \mathcal{B}$, by the previous claims (a) and (c); moreover, by (2), we have that

$$0 = \mathrm{Im}\Big(\varsigma\big(\psi(\pi(a)) - \varphi(\pi(a))\big)\Big) = \psi(\pi(\mathrm{Im}(\varsigma a))) - \varphi(\pi(\mathrm{Im}(\varsigma a)))$$

and hence

$$\begin{aligned}\varsigma(\psi(\pi(a)) - \varphi(\pi(a))) &= \big[\psi(\pi(\mathrm{Re}(\varsigma a))) - \varphi(\pi(\mathrm{Re}(\varsigma a)))\big] \\ &\quad + i\underbrace{\big[\psi(\pi(\mathrm{Im}(\varsigma a))) - \varphi(\pi(\mathrm{Im}(\varsigma a)))\big]}_{=0} \\ &= \psi(\pi(\tilde{a})) - \varphi(\pi(\tilde{a})),\end{aligned}$$

which gives

$$d_{\mathcal{D}}^{sa}(\varphi,\psi) \geq |\psi(\pi(\tilde{a})) - \varphi(\pi(\tilde{a}))|$$
$$= |\psi(\pi(a)) - \varphi(\pi(a))| \overset{(1)}{\geq} d_{\mathcal{D}}(\varphi,\psi) - \varepsilon.$$

Because $\varepsilon > 0$ was chosen arbitrarily, the claim follows.

□

Exercise 2:

Claim: ad_U is isometric.

Proof: $\|ad_U(x)\| = \|UxU^*\| \leq \|x\| \quad \forall x \in B(\mathcal{H}_1)$

$\|ad_{U^*}(y)\| = \|U^*yU\| \leq \|y\| \quad \forall y \in B(\mathcal{H}_2)$

and ad_U is invertible with $ad_U^{-1} = ad_{U^*}$;

thus, for all $x \in B(\mathcal{H}_1)$

$\|x\| = \|ad_{U^*}(ad_U(x))\| \leq \|ad_U(x)\| \leq \|x\|$

$\Rightarrow \|ad_U(x)\| = \|x\|$

□

Note that ad_U is obviously an injective $*$-homomorphism; hence, the claim also follows from Proposition 4.10, FA II.

Claim: $ad_U(A_1) = A_2$

Proof: For all $a \in \mathcal{A}_1$, we have that

$ad_U(\pi_1(a)) = U\pi_1(a)U^* = \pi_2(\Phi(a)) \in A_2$;

thus, by continuity $ad_U(A_1) \subseteq A_2$. Since at the same

time $U^* \pi_2(a) U = \pi_1(\Phi^{-1}(a))$ for all $a \in \mathcal{A}_2$,
we get analogously that $ad_{U^*}(A_2) \subseteq A_1$.

Hence $\quad A_2 = ad_U(ad_{U^*}(A_2)) \subseteq ad_U(A_1) \subseteq A_2$

$$\Rightarrow \quad ad_U(A_1) = A_2 \qquad \square$$

Claim: $\forall\, \varphi, \psi \in S(A_2)$: $d_{\mathcal{D}_1}(ad_U^* \varphi, ad_U^* \psi) = d_{\mathcal{D}_2}(\varphi, \psi)$

Proof: We put $\mathcal{B}_j := \{ a \in \mathcal{A}_j \mid \|[\mathcal{D}_j, \pi_j(a)]\| \leq 1 \}$ and check

$$\begin{aligned}
& d_{\mathcal{D}_1}(ad_U^* \varphi, ad_U^* \psi) \\
&= \sup \{ |(ad_U^* \psi)(\pi_1(a)) - (ad_U^* \varphi)(\pi_1(a))| \mid a \in \mathcal{B}_1 \} \\
&= \sup \{ |\psi(ad_U(\pi_1(a))) - \varphi(ad_U(\pi_1(a)))| \mid a \in \mathcal{B}_1 \} \\
&= \sup \{ |\psi(\pi_2(\Phi(a))) - \varphi(\pi_2(\Phi(a)))| \mid a \in \mathcal{B}_1 \} \\
&= \sup \{ |\psi(\pi_2(a)) - \varphi(\pi_2(a))| \mid a \in \Phi(\mathcal{B}_1) \} \\
&= d_{\mathcal{D}_2}(\varphi, \psi),
\end{aligned}$$

where we used that $\Phi(\mathcal{B}_1) = \mathcal{B}_2$. To see this, note that for $a \in \mathcal{A}_1$

$$\begin{aligned}
[\mathcal{D}_2, \pi_2(\Phi(a))] &= [\mathcal{D}_2, U \pi_1(a) U^*] \\
&= U [U^* \mathcal{D}_2 U, \pi_1(a)] U^* \\
&= ad_U([\mathcal{D}_1, \pi_1(a)])
\end{aligned}$$

and hence $\| [\mathcal{D}_2, \pi_2(\Phi(a))] \| = \| [\mathcal{D}_1, \pi_1(a)] \|$;

thus $\Phi(\mathcal{B}_1) \subseteq \mathcal{B}_2$. Repeating the same reasoning for Φ^{-1}, we get $\Phi(\mathcal{B}_1) = \mathcal{B}_2$. $\square$

Exercise 3:

(i) Since $\mathcal{H}$ is finite dimensional, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is trivially a spectral triple.

For $a = (a_1, a_2) \in \mathcal{A}$, we get

$$\mathcal{D}\pi(a) = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} \begin{pmatrix} a_1 \, id_{\mathcal{H}_1} & 0 \\ 0 & a_2 \, id_{\mathcal{H}_2} \end{pmatrix} = \begin{pmatrix} 0 & a_2 M^* \\ a_1 M & 0 \end{pmatrix}$$

$$\pi(a)\mathcal{D} = \begin{pmatrix} a_1 \, id_{\mathcal{H}_1} & 0 \\ 0 & a_2 \, id_{\mathcal{H}_2} \end{pmatrix} \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 M^* \\ a_2 M & 0 \end{pmatrix}$$

and thus

$$[\mathcal{D}, \pi(a)] = (a_2 - a_1) \begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix},$$

which gives $\| [\mathcal{D}, \pi(a)] \| = |a_2 - a_1| \, \| M \|$, since

$$X := \begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix}, \text{ satisfies}$$

$$X^* X = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix} \begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix} = \begin{pmatrix} M^* M & 0 \\ 0 & M M^* \end{pmatrix}$$

and hence $\| X \|^2 = \| X^* X \| = \| M^* M \| = \| M \|^2$,

as $\| M^* M \| = \| M \|^2 = \| M M^* \|$.

(ii) $d_{\mathcal{D}}(\delta_1, \delta_2) = \sup\{ |a_2 - a_1| \mid a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\| \leq 1 \}$

$$= \begin{cases} \frac{1}{\|M\|}, & \text{if } M \neq 0 \\ \infty, & \text{if } M = 0 \end{cases} \quad \text{by (i).}$$

(iii) Define $\Gamma = \begin{pmatrix} id_{\mathcal{H}_1} & 0 \\ 0 & -id_{\mathcal{H}_2} \end{pmatrix}$. Then clearly

$\Gamma = \Gamma^*$, $\Gamma^2 = id_{\mathcal{H}}$ and $\pi(a)\Gamma = \Gamma\pi(a)$ $\forall a \in \mathcal{A}$.

Moreover,

$$\mathcal{D}\Gamma = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}\begin{pmatrix} id_{\mathcal{H}_1} & 0 \\ 0 & -id_{\mathcal{H}_2} \end{pmatrix} = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix},$$

$$\Gamma\mathcal{D} = \begin{pmatrix} id_{\mathcal{H}_1} & 0 \\ 0 & -id_{\mathcal{H}_2} \end{pmatrix}\begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} = \begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix},$$

so that $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$, as desired.