Assignment 5A

Exercise 1:

We will prove that

$$\sigma_N(T) \overset{①}{\geq} \sup\{\|TP\|_1 \mid P \in B(\mathcal{H}) \text{ proj.}, \dim(P\mathcal{H}) = N\}$$
$$\overset{②}{\geq} \sup\{\operatorname{Tr}(P|T|P) \mid P \in B(\mathcal{H}) \text{ proj}, \dim(P\mathcal{H}) = N\}$$
$$\overset{③}{\geq} \sigma_N(T);$$

besides the formula for $\sigma_N(T)$ asserted in the exercise, this also shows that

$$\sigma_N(T) = \sup\{\operatorname{Tr}(P|T|P) \mid P \in B(\mathcal{H}) \text{ proj}, \dim(P\mathcal{H}) = N\}.$$

① By definition, we have that

$$\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n(T) \qquad \text{with}$$

$$\mu_n(T) = \inf\{\|T-S\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran}(S) \leq n\}$$

Claim: For all $T \in K(\mathcal{H})$ and each orthogonal projection $P \in B(\mathcal{H})$ with $\dim(P\mathcal{H}) = N$, we have that

$$\mu_n(TP) \leq \mu_n(T) \quad \forall n \geq 0$$

and $\mu_n(TP) = 0$ for all $n \geq N$.

Proof: For each $n \geq 0$, we have that

$$\mu_n(TP) = \inf\{\|TP - S\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran}(S) \leq n\}$$

$$\leq \inf\{\|TP - SP\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran}(S) \leq n\}$$

(since $\operatorname{ran}(SP) \subseteq \operatorname{ran}(S)$ and hence $\dim \operatorname{ran}(SP) \leq n$)

$$\leq \inf\{\|T - S\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran}(S) \leq n\}$$

(since $\|TP - SP\| = \|(T-S)P\| \leq \|T - S\|$)

$$= \mu_n(T).$$

Moreover, since $S := TP$ satisfies $\dim \operatorname{ran}(S) \leq N$, we conclude that $\mu_n(TP) = 0$ for all $n \geq N$. $\square$

Thus, we get

$$\|TP\|_1 = \operatorname{Tr}(|TP|) = \sum_{n=0}^{\infty} \mu_n(|TP|) = \sum_{n=0}^{\infty} \mu_n(TP)$$

$$= \sum_{n=0}^{N-1} \mu_n(TP) \leq \sum_{n=0}^{N-1} \mu_n(T) = \sigma_N(T),$$

which proves ①.

② Note that (since $P \leq 1$)

$$|TP|^2 = PT^*TP = P|T|1|T|P \geq (P|T|P)^2$$

and hence $|TP| \geq P|T|P$. Because Tr is positive, we conclude that

$$\|TP\|_1 = \mathrm{Tr}(|TP|) \geq \mathrm{Tr}(P|T|P),$$

which verifies ②.

③ Let $(\xi_n)_{n=0}^{\infty}$ be the orthonormal system of eigenvectors of $|T|$ with eigenvalues $(\mu_n(T))_{n=0}^{\infty}$. Further, let P be the orthogonal projection onto $\mathrm{span}\{\xi_n \mid 0 \leq n \leq N-1\}$. Then, since $(\xi_n)_{n=0}^{\infty}$ can be extended to an orthonormal basis of $\mathcal{H}$, we get that

$$\begin{aligned}
\mathrm{Tr}(P|T|P) &\geq \sum_{n=0}^{\infty} \langle P|T|P\xi_n, \xi_n \rangle \\
&= \sum_{n=0}^{\infty} \langle |T|P\xi_n, P\xi_n \rangle \\
&= \sum_{n=0}^{N-1} \langle \underbrace{|T|\xi_n}_{=\mu_n(T)\xi_n}, \xi_n \rangle = \sum_{n=0}^{N-1} \mu_n(T) = \sigma_N(T),
\end{aligned}$$

which proves ③.

Exercise 2

(i) • Take an orthogonal projection $P \in B(\mathcal{H})$ with $\dim(P\mathcal{H}) = N$.

Then

$$\|(T_1 + T_2)P\|_1 \leq \|T_1 P\|_1 + \|T_2 P\|_1$$

$$\overset{\text{Ex 1}}{\leq} \sigma_N(T_1) + \sigma_N(T_2).$$

Hence

$$\sigma_N(T_1 + T_2) = \sup\left\{ \|(T_1+T_2)P\|_1 \;\middle|\; \begin{array}{l} P \in B(\mathcal{H}) \text{ orth. proj.} \\ \dim(P\mathcal{H}) = N \end{array} \right\}$$

$$\leq \sigma_N(T_1) + \sigma_N(T_N)$$

• In other words:

$$\sigma_N : K(\mathcal{H}) \to [0, \infty), \quad T \mapsto \sigma_N(T)$$

satisfies the triangle inequality. Moreover, it follows from Exercise 1 that

$$\sigma_N(\lambda T) = |\lambda| \, \sigma_N(T) \quad \text{for all } \lambda \in \mathbb{C}.$$

Finally, if $\sigma_N(T) = 0$ for some $T \in K(\mathcal{H})$, it follows that $TP = 0$ for each orthogonal projection $P \in B(\mathcal{H})$ with $\dim(P\mathcal{H}) = N$ and hence $T|_V = 0$ for each subspace $V \subseteq \mathcal{H}$ with $\dim V = N$; therefore $T = 0$, since each $\xi \in \mathcal{H}$ sits in such a space V as $N \geq 1$.

$\Rightarrow$ σ_N is a norm on $K(\mathcal{H})$

(ii) Let $\varepsilon > 0$ be given. From Exercise 1, it follows that there are projections $P_1, P_2 \in B(\mathcal{H})$ such that

- $\mathrm{Tr}(P_1 T_1 P_1) \geq \sigma_N(T_1) - \varepsilon$, $\dim(P_1 \mathcal{H}) = N$,
- $\mathrm{Tr}(P_2 T_2 P_2) \geq \sigma_N(T_2) - \varepsilon$, $\dim(P_2 \mathcal{H}) = N$.

Note that $\dim(P_1\mathcal{H} + P_2\mathcal{H}) \leq 2N$; thus, we find a subspace $V \subseteq \mathcal{H}$ with $\dim V = 2N$ and $P_1\mathcal{H} + P_2\mathcal{H} \subseteq V$. Let P be the orthogonal projection onto V. Then, by Exercise 1, we conclude that

$$
\begin{aligned}
\sigma_{2N}(T_1 + T_2) &\geq \mathrm{Tr}(P(T_1 + T_2)P) \\
&= \mathrm{Tr}(PT_1P) + \mathrm{Tr}(PT_2P) \\
&\overset{(*)}{\geq} \mathrm{Tr}(P_1 T_1 P_1) + \mathrm{Tr}(P_2 T_2 P_2) \\
&\geq (\sigma_N(T_1) + \sigma_N(T_2)) - \varepsilon .
\end{aligned}
$$

For $(*)$, note that

$$
\begin{aligned}
\mathrm{Tr}(PT_1P) &= \mathrm{Tr}(T_1^{1/2} P T_1^{1/2}) \\
&\geq \mathrm{Tr}(T_1^{1/2} P_1 T_1^{1/2}) = \mathrm{Tr}(P_1 T_1 P_1)
\end{aligned}
$$

and similarly $\mathrm{Tr}(PT_2P) \geq \mathrm{Tr}(P_2 T_2 P_2)$.

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\sigma_{2N}(T_1 + T_2) \geq \sigma_N(T_1) + \sigma_N(T_2).$$

(iii) From (i), we directly get that

$$\gamma_N(T_1+T_2) \leq \gamma_N(T_1) + \gamma_N(T_2).$$

From (ii), it follows that

$$\gamma_N(T_1) + \gamma_N(T_2) = \frac{1}{\log(N)}\left(\sigma_N(T_1) + \sigma_N(T_2)\right)$$

$$\leq \frac{1}{\log(N)}\,\sigma_{2N}(T_1+T_2) = \frac{\log(2N)}{\log(N)}\,\gamma_{2N}(T_1+T_2)$$

$$= \left(1 + \frac{\log(2)}{\log(N)}\right)\gamma_{2N}(T_1+T_2).$$