Exercise 1:

We will prove that

\[
\sigma_N(T) \geq \sup \{ \|TP\|, \; P \in \mathcal{B}(\mathcal{H}) \text{ proj, dim } (\mathcal{P}\mathcal{H}) = N \}
\]

\[
\geq \sup \{ \text{Tr} (PPTP), \; P \in \mathcal{B}(\mathcal{H}) \text{ proj, dim } (\mathcal{P}\mathcal{H}) = N \}
\]

\[
\geq \sigma_N(T);
\]

Besides the formula for \(\sigma_N(T)\) asserted in the exercise, this also shows that

\[
\sigma_N(T) = \sup \{ \text{Tr} (PPTP), \; P \in \mathcal{B}(\mathcal{H}) \text{ proj, dim } (\mathcal{P}\mathcal{H}) = N \}.
\]

\(1\) By definition, we have that

\[
\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n(T) \quad \text{with}
\]

\[
\mu_n(T) = \inf \{ \|T - S\|, \; S \in \mathcal{B}(\mathcal{H}) : \text{dim ran}(S) \leq n \}.
\]

Claim: For all \(T \in \mathcal{K}(\mathcal{H})\) and each orthogonal projection \(P \in \mathcal{B}(\mathcal{H}) \text{ with dim } (\mathcal{P}\mathcal{H}) = N\), we have that

\[
\mu_n(TP) \leq \mu_n(T), \; \forall n \geq 0
\]

and \(\mu_n(TP) = 0\) for all \(n \geq N\).
Proof: For each $n \geq 0$, we have that

$$\mu_n(TP) = \inf \{ \|TP - S\| \mid S \in \mathcal{B}(\mathcal{H}) : \text{dim } \text{ran}(S) \leq n \}$$

$$\leq \inf \{ \|TP - SP\| \mid S \in \mathcal{B}(\mathcal{H}) : \text{dim } \text{ran}(S) \leq n \}$$

(since $\text{ran}(SP) \subseteq \text{ran}(S)$ and hence $\text{dim } \text{ran}(SP) \leq n$)

$$\leq \inf \{ \|T - S\| \mid S \in \mathcal{B}(\mathcal{H}) : \text{dim } \text{ran}(S) \leq n \}$$

(since $\|TP - SP\| = \|T(S)P\| \leq \|T - S\|$)

$$= \mu_n(T).$$

Moreover, since $S := TP$ satisfies $\text{dim } \text{ran}(S) \leq N$, we conclude that $\mu_n(TP) = 0$ for all $n \geq N$. □

Thus, we get

$$\|TP\|_1 = \text{Tr}(\|TP\|_1) = \sum_{n=0}^{\infty} \mu_n(\|TP\|_1) = \sum_{n=0}^{\infty} \mu_n(TP)$$

$$= \sum_{n=0}^{N-1} \mu_n(TP) \leq \sum_{n=0}^{N-1} \mu_n(T) = \sigma_n(T),$$

which proves (1).

(2) Note that (since $P \leq 1$)

$$\|TP\|^2 = PT^*TP = P(1(PTP)P) \leq (P(PTP))^2$$

and hence $\|TP\| \geq P(PTP)$. Because $\text{Tr}$ is positive,
we conclude that

\[ \|TP\|_a = \text{Tr} (TP1) \geq \text{Tr} (P1TP), \]

which verifies (2).

(3) Let \((\beta_n)_{n=0}^\infty\) be the orthonormal system of eigenvectors of \(TT1\) with eigenvalues \((\mu_n(T))_{n=0}^\infty\). Further, let \(P\) be the orthogonal projection onto span \(\{\beta_n | 1 \leq n \leq N-1\}\).

Then, since \((\beta_n)_{n=0}^\infty\) can be extended to an orthonormal basis of \(H\), we get that

\[ \text{Tr} (P1TP) \geq \sum_{n=0}^\infty \langle P1TP \beta_n, \beta_n \rangle \]

\[ = \sum_{n=0}^\infty \langle TT1P \beta_n, P \beta_n \rangle \]

\[ = \sum_{n=0}^{N-1} \langle TT1 \beta_n, \beta_n \rangle = \sum_{n=0}^{N-1} \mu_n(T) = \sigma_N(T), \]

which proves (3).
(i) Take an orthogonal projection $P \in B(H)$ with $\dim(PH) = N$. Then

\[ \| (T_1 + T_2) P \|_1 \leq \| T_1 P \|_1 + \| T_2 P \|_1 \]

\[ \leq \sigma_N(T_1) + \sigma_N(T_2). \]

Hence

\[ \sigma_N(T_1 + T_2) = \sup \left\{ \| (T_1 + T_2) P \|_1 \mid P \in B(H) \text{ orth. proj.} \right\} \]

\[ \leq \sigma_N(T_1) + \sigma_N(T_2) \]

• In other words:

\[ \sigma_N : \mathcal{K}(H) \to [0, \infty), \ T \mapsto \sigma_N(T) \]

satisfies the triangle inequality. Moreover, it follows from Exercise 1 that

\[ \sigma_N(\lambda T) = |\lambda| \sigma_N(T) \quad \text{for all } \lambda \in \mathbb{C}. \]

Finally, if $\sigma_N(T) = 0$ for some $T \in \mathcal{K}(H)$, it follows that $TP = 0$ for each orthogonal projection $P \in B(H)$ with $\dim(PH) = N$ and hence $T|_V = 0$ for each subspace $V \in H$ with $\dim V = N$; therefore $T = 0$, since each $\mathcal{J} \in H$ is in such a manner $V$ as $N \geq 1$.

$\Rightarrow \sigma_N$ is a norm on $\mathcal{K}(H)$.
(ii) Let $\varepsilon > 0$ be given. From Exercise 1, it follows that there are projections $P_1, P_2 \in \mathcal{B}(H)$ such that

\begin{itemize}
  \item $\text{Tr} (P_1T_1P_1) \geq \sigma_N (T_1) - \varepsilon$, $\dim (P_1H) = N$,
  \item $\text{Tr} (P_2T_2P_2) \geq \sigma_N (T_2) - \varepsilon$, $\dim (P_2H) = N$.
\end{itemize}

Note that $\dim (P_1H + P_2H) \leq 2N$; thus, we find a subspace $V \subseteq H$ with $\dim V = 2N$ and $P_1H + P_2H \subseteq V$. Let $P$ be the orthogonal projection onto $V$. Then, by Exercise 1, we conclude that

$$
\sigma_{2N} (T_1 + T_2) \geq \text{Tr} (P (T_1 + T_2) P) \\
= \text{Tr} (PT_1P) + \text{Tr} (PT_2P) \\
\geq \text{Tr} (P_1T_1P_1) + \text{Tr} (P_2T_2P_2) \\
\geq (\sigma_N (T_1) + \sigma_N (T_2)) - \varepsilon.
$$

(\text{For (x), note that})

$$
\text{Tr} (PT_1P) = \text{Tr} (T_1^{1/2} P T_1^{1/2}) \\
\geq \text{Tr} (T_1^{1/2} P_1 T_1^{1/2}) = \text{Tr} (P_1T_1P_1)
$$

and similarly $\text{Tr} (PT_2P) \geq \text{Tr} (P_2T_2P_2)$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$
\sigma_{2N} (T_1 + T_2) \geq \sigma_N (T_1) + \sigma_N (T_2).
$$
(iii) From (i), we directly get that

\[ \gamma_N(T_1 + T_2) \leq \gamma_N(T_1) + \gamma_N(T_2). \]

From (ii), it follows that

\[ \gamma_N(T_1) + \gamma_N(T_2) = \frac{1}{\text{loge}(N)} \left( \sigma_N(T_1) + \sigma_N(T_2) \right) \]

\[ \leq \frac{1}{\text{loge}(N)} \sigma_{2N}(T_1 + T_2) = \frac{\text{loge}(2N)}{\text{loge}(N)} \gamma_{2N}(T_1 + T_2) \]

\[ = \left( 1 + \frac{\text{loge}(2)}{\text{loge}(N)} \right) \gamma_{2N}(T_1 + T_2). \]