UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Introduction to Noncommutative Differential Geometry
Summer term 2019

Assignment 5B
for the tutorial on *Tuesday, June 25 (!), 10:15 am* (in Seminar Room 10)

Note once more that there will be ***no lecture*** *on Monday, June 17, and* ***no problem session*** *on Tuesday, June 18; the* ***next problem session*** *is thus postponed to Tuesday, June 25, and the* ***next lecture*** *takes place on Monday, June 24.*

Exercise 1. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space.

(i) Let $T \in K(\mathcal{H})$ and $N \in \mathbb{N}$ be given. Prove the formula
$$\sigma_N(T) = \inf\big\{\|R\|_1 + N\|S\| \;\big|\; R \in \mathcal{L}^1(\mathcal{H}), S \in K(\mathcal{H})\colon T = R + S\big\}$$
for the value $\sigma_N(T)$ that was defined in Definition 4.5 of the lecture.

(ii) Like in Remark 4.10 (ii), we define for every $T \in K(\mathcal{H})$ and each $\lambda \geq 0$
$$\sigma_\lambda(T) := \inf\big\{\|R\|_1 + \lambda\|S\| \;\big|\; R \in \mathcal{L}^1(\mathcal{H}), S \in K(\mathcal{H})\colon T = R + S\big\}.$$
Due to (i), this interpolates the values $\sigma_N(T)$. Show that this interpolation is in fact piecewise linear, i.e., prove that $\sigma_\lambda(T) = \lambda\|T\|$ holds for every $\lambda \in [0,1)$ and that
$$\sigma_{N+\lambda}(T) = (1-\lambda)\sigma_N(T) + \lambda\sigma_{N+1}(T)$$
holds for each $N \in \mathbb{N}$ and every $\lambda \in [0,1)$.

Exercise 2. Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples with infinite dimensional separable complex Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and suppose that $\Gamma_1 \in B(\mathcal{H}_1)$ is a grading on $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$. Put
$$\mathcal{A} := \mathcal{A}_1 \otimes_\mathbb{C} \mathcal{A}_2, \qquad \mathcal{H} := \mathcal{H}_1 \mathbin{\overline{\otimes}_\mathbb{C}} \mathcal{H}_2, \qquad \text{and} \qquad \mathcal{D} := \mathcal{D}_1 \otimes \mathrm{id}_{\mathcal{H}_2} + \Gamma_1 \otimes \mathcal{D}_2.$$
Prove the following assertions:

(i) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple.

(ii) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable[1] whenever both of the spectral triples $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ are θ-summable.

[1] Recall from Definition 4.11 that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to be *θ-summable* if $e^{-t\mathcal{D}^2} \in \mathcal{L}^1(\mathcal{H})$ for each $t > 0$.