Assignment 5B

Exercise 1:

(i) "$\leq$": For any decomposition $T = R + S$ with $R \in \mathcal{L}^1(\mathcal{H})$ and $S \in \mathcal{K}(\mathcal{H})$, we get from Exercise 5A-2(i) that

$$\sigma_N(T) = \sigma_N(R+S) \leq \sigma_N(R) + \sigma_N(S).$$

Since

- $\mu_n(S) \leq \mu_0(S) = \|S\| \quad \forall n \geq 0$
- $\sigma_N(R) = \sum_{n=0}^{N-1} \mu_n(|R|) \leq \sum_{n=0}^{\infty} \mu_n(|R|) = \mathrm{Tr}(|R|) = \|R\|_1$

we get that

$$\sigma_N(T) \leq \|R\|_1 + N\|S\|$$

"$\geq$": Let P_N be the orthogonal projection onto $\mathrm{span}\{\xi_n \mid 0 \leq n \leq N-1\}$, where $(\xi_n)_{n=0}^{\infty}$ are the eigenvectors of $|T|$ associated to $(\mu_n(T))_{n=0}^{\infty}$.

Put
- $R := (|T| - \mu_N(T))P_N \in \mathcal{L}^1(\mathcal{H})$,
- $S := \mu_N(T)P_N + |T|(1-P_N) \in \mathcal{K}(\mathcal{H})$.

Then $|T| = R + S$, where $\|S\| = \mu_N(T)$ and

$$\|R\|_1 = \sum_{n=0}^{N-1} \big(\mu_n(T) - \mu_N(T)\big) = \sigma_N(T) - N\mu_N(T).$$

Thus,

$$\|R\|_1 + N\|S\| = (\sigma_N(T) - N\mu_N(T)) + N\mu_N(T) = \sigma_N(T).$$

(ii) <u>Claim</u>: $\forall \lambda \in [0,1)$: $\sigma_\lambda(T) = \lambda \|T\|$.

<u>Proof</u>: "$\leq$": Take $(R,S) = (0,T)$; this gives

$$\sigma_\lambda(T) \leq \|R\|_1 + \lambda\|S\| = \lambda\|T\|.$$

"$\geq$": For any decomposition $T = R+S$, we have (since $\|R\|_1 \geq \mu_0(R) = \|R\|$)

$$\begin{aligned}\|R\|_1 + \lambda\|S\| &\geq \|R\| + \lambda\|S\| \\ &\geq \lambda(\|R\| + \|S\|) \\ &\geq \lambda\|R+S\| = \lambda\|T\|;\end{aligned}$$

thus $\sigma_\lambda(T) \geq \lambda\|T\|$. $\square$

<u>Claim</u>: $\forall N \in \mathbb{N}\ \forall \lambda \in [0,1)$:

$$\sigma_{N+\lambda}(T) = (1-\lambda)\sigma_N(T) + \lambda\sigma_{N+1}(T).$$

<u>Proof</u>: "$\leq$": Take (R,S) like in Exercise 1(i); then

$$\begin{aligned}\sigma_{N+\lambda}(T) &\leq \|R\|_1 + (N+\lambda)\|S\| \\ &\leq (\sigma_N(T) - N\mu_N(T)) + (N+\lambda)\mu_N(T) \\ &= \sigma_N(T) + \lambda\mu_N(T)\end{aligned}$$

$$= (1-\lambda)\sigma_N(T) + \lambda(\sigma_N(T) + \mu_N(T))$$

$$= (1-\lambda)\sigma_N(T) + \lambda\,\sigma_{N+1}(T)$$

<u>" $\geq$ "</u>: For each decomposition $T = R + S$, we have

$$\|R\|_1 + (N+\lambda)\|S\|$$

$$= (1-\lambda)\big(\underbrace{\|R\|_1 + N\|S\|}_{\geq\,\sigma_N(T)}\big) + \lambda\big(\underbrace{\|R\|_1 + (N+1)\|S\|}_{\geq\,\sigma_{N+1}(T)}\big)$$

$(1-\lambda)N + \lambda(N+1) = N + \lambda$

$$\geq (1-\lambda)\sigma_N(T) + \lambda\,\sigma_{N+1}(T)$$

□

<u>Exercise 2:</u>

(i) <u>Claim:</u> $\mathcal{D} := \mathcal{D}_1 \otimes id_{\mathcal{H}_2} + \Gamma_1 \otimes \mathcal{D}_2$ with

$\operatorname{dom}\mathcal{D} := \operatorname{dom}\mathcal{D}_1 \otimes \operatorname{dom}\mathcal{D}_2 \subseteq \mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$

is closable and essentially selfadjoint.

(In the following, we thus identify $\mathcal{D}$ with its closure.)

<u>Step 1:</u> diagonalization of $\mathcal{D}$

- Since $(\mathcal{A}_i, \mathcal{H}_i, \mathcal{D}_i)$ is a spectral triple, $i=1,2$, we find an orthonormal basis of $\mathcal{H}_i$ of the form

$$\{\xi^i_{\lambda,m} \mid \lambda \in \sigma(\mathcal{D}_i)\setminus\{0\},\ m = 1,\dots,M^i_\lambda\} \cup \{\gamma^i_n \mid n = 1,\dots,N\},$$

where M^i_λ is the (finite) multiplicity of each eigenvalue

$\lambda \in \sigma(D_i)\setminus\{0\}$ and $\{\gamma_n^i \mid n=1,\dots,N^i\}$ is an orthonormal basis of $\ker D_i$; note that $N^i = \dim \ker D_i$ is finite. (Recall that D_i has pure point spectrum and the only limit point of $\{|\lambda| \mid \lambda \in \sigma(D_i)\}$ is ∞.)

We have that

$$D_i \xi^i_{\lambda,m} = \lambda \xi^i_{\lambda,m} \quad \text{for } \lambda \in \sigma(D_i)\setminus\{0\},\ m=1,\dots,M^i_\lambda,$$

$$D_i \gamma^i_n = 0 \quad \text{for } n=1,\dots,N^i.$$

- Since $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ is even with grading Γ_1, we have that for each $\lambda \in \sigma(D_1)\setminus\{0\}$ also $-\lambda \in \sigma(D_1)\setminus\{0\}$.

  Indeed, if $\lambda \in \sigma(D_1)\setminus\{0\}$ with eigenvector $0 \neq \xi \in \mathcal{H}_1$, then
  $$D_1(\Gamma_1 \xi) = -\Gamma_1 D_1 \xi = -\Gamma_1(\lambda \xi) = -\lambda \Gamma_1 \xi,$$
  i.e., $-\lambda \in \sigma(D_1)\setminus\{0\}$ with eigenvector $\Gamma_1 \xi$ (which is non-zero as Γ_1 is invertible). Thus, $M^1_\lambda = M^1_{-\lambda}$ and we may arrange the orthonormal basis of $\mathcal{H}_1$ such that $\Gamma_1 \xi^1_{\lambda,m} = \xi^1_{-\lambda,m}$.

  Furthermore, Γ_1 maps $\ker D_1$ to itself; indeed, for $\gamma \in \ker D_1$, we have that $D_1(\Gamma_1\gamma) = -\Gamma_1(D_1\gamma) = 0$. Thus, we may suppose that $\{\gamma^1_n \mid n=1,\dots,N^1\}$ consists of eigenvectors of Γ_1, say $\Gamma_1 \gamma^1_n = \mu_n \gamma^1_n$, $\mu_n \in \{\pm 1\}$.

- Consider $\{\xi^1_{\lambda,m} \otimes \gamma^2_n \mid \lambda \in \sigma(D_1)\setminus\{0\},\ m=1,\dots,M^1_\lambda,\ n=1,\dots,N^2\}$.

This forms an orthonormal system of eigenvectors of $\mathcal{D}$ with $\mathcal{D}(\xi^1_{\lambda,m} \otimes \gamma^2_n) = \lambda\, \xi^1_{\lambda,m} \otimes \gamma_n$.

Similarly, $\{\gamma^1_n \otimes \xi^2_{\lambda,m} \mid \lambda \in \sigma(\mathcal{D}_2)\setminus\{0\},\, m = 1,\dots,M^2_\lambda,\, n = 1,\dots,N^1\}$ forms an orthonormal system of eigenvectors of $\mathcal{D}$ with $\mathcal{D}(\gamma^1_n \otimes \xi^2_{\lambda,m}) = \mu_n \lambda\, \gamma^1_n \otimes \xi_{\lambda,m}$.

Furthermore, $\{\gamma^1_{n_1} \otimes \gamma^2_{n_2} \mid n_1 = 1,\dots,N^1,\, n_2 = 1,\dots,N^2\}$ gives an orthonormal system in $\ker \mathcal{D}$.

Finally, we note that

$$\mathcal{D}(\xi^1_{\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2}) = \lambda_1\, \xi^1_{\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2} + \lambda_2 \underbrace{\Gamma_1 \xi^1_{\lambda_1,m_1}}_{= \xi^1_{-\lambda_1,m_1}} \otimes \xi^2_{\lambda_2,m_2},$$

$$\mathcal{D}(\xi^1_{-\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2}) = -\lambda_1\, \xi^1_{-\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2} + \lambda_2 \underbrace{\Gamma_1 \xi^1_{-\lambda_1,m_1}}_{= \xi^1_{\lambda_1,m_1}} \otimes \xi^2_{\lambda_2,m_2},$$

Thus, restricted to the span of $\{\xi^1_{\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2},\, \xi^1_{-\lambda_1,m_1} \otimes \xi^2_{\lambda_2,m_2}\}$,

$$\mathcal{D} \mathrel{\hat{=}} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & -\lambda_1 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{\lambda_1^2 + \lambda_2^2}$ with eigenvectors

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad \text{for } \theta = \frac{1}{2}\arctan\left(\frac{\lambda_2}{\lambda_1}\right);$$

to verify the latter, one uses that $\tan(\theta) = \dfrac{\tan(2\theta)}{1 + \sqrt{1 + \tan^2(2\theta)}}$.

Thus, an orthonormal system of eigenvectors of D (with the eigenvalues $\pm\sqrt{\lambda_1^2+\lambda_2^2}$) is given by

$$\left(\cos(\Theta)\, \xi^1_{\lambda_1, m_1} + \sin(\Theta)\, \xi^1_{-\lambda_1, m_1}\right) \otimes \xi^2_{\lambda_2, m_2},$$

$$\left(-\sin(\Theta)\, \xi^1_{\lambda_1, m_1} + \cos(\Theta)\, \xi^1_{-\lambda_1, m_1}\right) \otimes \xi^2_{\lambda_2, m_2}$$

for $\Theta = \frac{1}{2} \arctan\left(\frac{\lambda_2}{\lambda_1}\right)$.

Because we began with orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, putting those orthonormal systems together, we obtain an orthonormal basis of $\mathcal{H}_1 \overline{\otimes}_{\mathbb{C}} \mathcal{H}_2$ which consists of eigenvectors of D.

<u>Step 2</u>: diagonal operators are essentially selfadjoint

Let $T: \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be a symmetric linear operator and suppose that there exists an orthonormal basis $(e_n)_{n=0}^{\infty}$ of $\mathcal{H}$ that consists of eigenvectors of T with associated eigenvalues $(\lambda_n)_{n=0}^{\infty}$ (in $\mathbb{R}$), i.e.

$$\forall n \in \mathbb{N}_0: \quad e_n \in \operatorname{dom} T, \quad T e_n = \lambda_n e_n.$$

Then T is essentially selfadjoint.

<u>Proof</u>:

Note that T is densely defined as $\operatorname{span}\{e_n \mid n \in \mathbb{N}_0\} \subseteq \operatorname{dom} T$.

It suffices to prove that $\operatorname{ran}(T \pm i)^{\perp} = \{0\}$.

For $\eta \in \operatorname{ran}(T \pm i)^\perp$, we have for all $n \in \mathbb{N}_0$

$$0 = \langle (T \pm i) e_n, \eta \rangle = \underbrace{(\lambda_n \pm i)}_{=0} \langle e_n, \eta \rangle ;$$

thus, $\eta = \sum_{n=0}^{\infty} \langle \eta, e_n \rangle e_n = 0.$ $\square$

Since D is symmetric and (see Step 1) diagonalizable in the sense of the previous statement, it follows that D is essentially selfadjoint.

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<u>Claim:</u> $(\mathcal{A}_1 \otimes_{\mathbb{C}} \mathcal{A}, \mathcal{H}_1 \bar{\otimes}_{\mathbb{C}} \mathcal{H}_2, \overline{D})$ is a spectral triple

The only questionable property is that $\overline{D}$ has compact resolvents. But as $\sigma(D)$ was computed explicitly, which shows that the only limit point of $\{|\lambda| \mid \lambda \in \sigma(D)\}$ is ∞, this is immediately clear.

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(ii) We observe that (by the properties of the grading Γ_1)

$$D^2 = D_1^2 \otimes id_{\mathcal{H}_2} + \underbrace{(\Gamma_1 D_1 + D_1 \Gamma_1)}_{=0} \otimes D_2 + \Gamma_1^2 \otimes D_2^2$$

$$= D_1^2 \otimes id_{\mathcal{H}_2} + id_{\mathcal{H}_1} \otimes D_2^2 .$$

Since $D_1^2 \otimes id_{\mathcal{H}_2}$ and $id_{\mathcal{H}_1} \otimes D_2^2$ commute, we get that

$$e^{-tD^2} = e^{-tD_1^2} \otimes e^{-tD_2^2} \quad \text{on } \operatorname{dom} D \text{ for all } t > 0.$$

Denote by

- $(\xi_n)_{n=0}^{\infty}$ an orthonormal basis of $\mathcal{H}_1$ in dom $\mathcal{D}_1$,
- $(\eta_m)_{m=0}^{\infty}$ an orthonormal basis of $\mathcal{H}_2$ in dom $\mathcal{D}_2$.

Then,

$$\mathrm{Tr}\left(e^{-t\overline{\mathcal{D}}^2}\right) = \sum_{n,m=0}^{\infty} \langle e^{-t\mathcal{D}^2} \xi_n \otimes \eta_m , \xi_n \otimes \eta_m \rangle$$

$$= \sum_{n,m=0}^{\infty} \langle e^{-t\mathcal{D}_1^2} \xi_n \otimes e^{-t\mathcal{D}_2^2} \eta_m , \xi_n \otimes \eta_m \rangle$$

$$= \sum_{n,m=0}^{\infty} \langle e^{-t\mathcal{D}_1^2} \xi_n , \xi_n \rangle \langle e^{-t\mathcal{D}_2^2} \eta_m , \eta_m \rangle$$

$$= \left(\sum_{n=0}^{\infty} \langle e^{-t\mathcal{D}_1^2} \xi_n , \xi_n \rangle \right) \left(\sum_{m=0}^{\infty} \langle e^{-t\mathcal{D}_2^2} \eta_m , \eta_m \rangle \right)$$

$$= \mathrm{Tr}\left(e^{-t\mathcal{D}_1^2}\right) \mathrm{Tr}\left(e^{-t\mathcal{D}_2^2}\right) < \infty$$

$\square$