UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Introduction to Noncommutative Differential Geometry
Summer term 2019

Assignment 6A
for the tutorial on *Tuesday, July 9, 10:15 am* (in Seminar Room 10)

We recall some basic facts about the Fourier transform on $\mathbb{R}^n$ which can be used for the solution of the following exercises without proof.

(i) Let λ^n be the Lebesgue measure on $\mathbb{R}^n$. For every function $u \in L^1(\mathbb{R}^n, \lambda^n)$, we define its *Fourier transform* $\hat{u} = \mathcal{F}u$ by

$$(\mathcal{F}u)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} u(x)\, d\lambda^n(x) \qquad \text{for each } \xi \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, i.e., we have $\langle \xi, x\rangle = \sum_{j=1}^n \xi_j x_j$ for each $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ in $\mathbb{R}^n$. It is known that $\mathcal{F}f \in C_0(\mathbb{R}^n)$.

(ii) Let $\mathcal{S}(\mathbb{R}^n)$ be the *Schwartz space*, i.e., the space of all smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^m) |(\partial^\alpha f)(x)| < \infty$$

for each $m \in \mathbb{N}_0$ and each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. The Fourier transform $\mathcal{F}$, if restricted to $\mathcal{S}(\mathbb{R}^n)$, has the remarkable property that it induces a bijection $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, with inverse given by

$$(\mathcal{F}^{-1}v)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} v(\xi)\, d\lambda^n(\xi) \qquad \text{for each } x \in \mathbb{R}^n.$$

Exercise 1. Consider the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Prove the following properties:

(i) For each $u \in \mathcal{S}(\mathbb{R}^n)$ and each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we have that

$$\partial^\alpha(\mathcal{F}u) = (-i)^{|\alpha|} \mathcal{F}(m_\alpha u),$$

where m_α denotes the function

$$m_\alpha : \ \mathbb{R}^n \to \mathbb{C}, \quad (x_1, \dots, x_n) \mapsto x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

(ii) For each $u \in \mathcal{S}(\mathbb{R}^n)$ and each multi-index $\alpha \in \mathbb{N}_0^n$, we have that

$$\mathcal{F}(\partial^\alpha u) = i^{|\alpha|} m_\alpha \mathcal{F}u.$$

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Exercise 2. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open. Consider a differential operator $P : C^\infty(\Omega) \to C^\infty(\Omega)$ which is of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha (-i)^{|\alpha|} \partial^\alpha$$

for some integer $m \geq 0$ and with coefficients $a_\alpha \in C^\infty(\Omega)$ for each $|\alpha| \leq m$. Let

$$p^P : \Omega \times \mathbb{R}^n \to \mathbb{R}, \quad (x, \xi) \mapsto \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

be the complete symbol of P. Prove that for each $u \in \mathcal{S}(\mathbb{R}^n)$ and every point $x \in \Omega$

$$(Pu|_\Omega)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} p^P(x, \xi) \hat{u}(\xi) \, d\lambda^n(\xi).$$