Assignment 6 A

Exercise 1:

(i) We check that for all $\xi, \eta \in \mathbb{R}^n$

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\Big((\mathcal{F}u)(\xi + \varepsilon\eta) - (\mathcal{F}u)(\xi) \Big)$$

$$= \lim_{\varepsilon \searrow 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1}{\varepsilon}\left(e^{-i<\xi+\varepsilon\eta, x>} - e^{-i<\xi, x>}\right) u(x)\, d\lambda^n(x)$$

$$= \lim_{\varepsilon \searrow 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \underbrace{\frac{1}{\varepsilon}\left(e^{-i\varepsilon<\eta, x>} - 1\right)}_{=: h_\varepsilon(x)} e^{-i<\xi, x>} u(x)\, d\lambda^n(x)$$

$$\overset{(*)}{=} \frac{-i}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i<\xi, x>} <\eta, x> u(x)\, d\lambda^n(x)$$

$$= -i\, \mathcal{F}(<\eta, \cdot> u),$$

where $(*)$ uses ~~Lebesgue's~~ dominated convergence theorem; we have to note that

- $h_\varepsilon \to h$ pointwise as $\varepsilon \searrow 0$, where $h(x) := -i<\eta, x>$
- $h_\varepsilon(x) = -i<\eta, x> \frac{1}{\varepsilon}\int_0^\varepsilon e^{-it<\eta, x>}\, dt$, so that $|h_\varepsilon(x)| \leq |<\eta, x>|$, and $<\eta, \cdot> u \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{L}^1(\mathbb{R}^n, \lambda^n)$.

In particular

$$\partial_j(\mathcal{F}u) = -i\, \mathcal{F}(m_{(0,\dots,\underset{\substack{\uparrow \\ j\text{th position}}}{1},\dots,0)}\, u) \qquad \forall j = 1,\dots,n$$

and by induction

$$\partial^\alpha(\mathcal{F}u) = (-i)^{|\alpha|}\, \mathcal{F}(m_\alpha u) \qquad \forall \alpha$$

(ii) We compute that

$$(\mathcal{F}(\partial_j u))(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \underbrace{e^{-i\langle \xi, x\rangle} (\partial_j u)(x)}_{\substack{= \partial_j (e^{-i\langle \xi, \cdot\rangle} u)(x) \\ + i \xi_j e^{-i\langle \xi, x\rangle} u(x)}} d\lambda^n(x)$$

$$= i \xi_j (\mathcal{F}u)(\xi),$$

where we used that

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_j (e^{-i\langle \xi, \cdot\rangle} u)(x)\, d\lambda^n(x) = 0. \qquad (**)$$

since $u \in \mathcal{S}(\mathbb{R}^n)$.

Hence,

$$\mathcal{F}(\partial_j u) = i\, m_{(0,\dots,\underset{\substack{\uparrow \\ j\text{th position}}}{1},\dots,0)}\, \mathcal{F}u \qquad \forall j = 1,\dots,n,$$

and by induction

$$\mathcal{F}(\partial^\alpha u) = i^{|\alpha|} m_\alpha \mathcal{F}u \qquad \forall \alpha .$$

Exercise 2:

We check that for $x \in \Omega$

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} p^P(x,\xi)\, \hat{u}(\xi)\, d\lambda^n(\xi)$$

$$= \sum_{|\alpha| \le m} a_\alpha(x)\, \mathcal{F}^{-1}(m_\alpha \hat{u})(x)$$

$$= \sum_{|\alpha| \le m} a_\alpha(x)\, (-i)^{|\alpha|} (\partial^\alpha u)(x) = (Pu|_\Omega)(x)$$

(**) can be seen as follows: $v := e^{-i\langle \xi, \cdot\rangle} u \in \mathcal{S}(\mathbb{R}^n)$

- $\int_{\mathbb{R}^n} (\partial_i v)(x)\, d\lambda^n(x) = \lim_{r\to\infty} \int_{Q_r} (\partial_i v)(x)\, d\lambda^n(x),$

  where $Q_r := \mathbb{R} \times \ldots \times \mathbb{R} \times [-r, r] \times \mathbb{R} \times \ldots \times \mathbb{R} \subseteq \mathbb{R}^n$,

  by the dominated convergence theorem

- $$\int_{Q_r} (\partial_i v)(x)\, d\lambda^n(x) = \int_{\mathbb{R}^{n-1}} \underbrace{\left(\int_{[-r,r]} (\partial_i v)(x)\, d\lambda^1(x_i) \right)}_{= v(x', r) - v(x', -r)} d\lambda^{n-1}(x')$$

  by Fubini's theorem

- $\int_{\mathbb{R}^{n-1}} v(x', a)\, d\lambda^{n-1}(x') \longrightarrow 0$ as $|a| \to \infty$ $\left(\begin{array}{l}\text{dom. conv.}\\ v \in \mathcal{S}(\mathbb{R}^n)\end{array}\right)$