Assignment 6B

Exercise 1:

We note that for each $t>0$

$$e^{-tD^2} = (1+D^2)^{P/2} e^{-tD^2} (1+D^2)^{-P/2},$$

where
- $(1+D^2)^{-P/2} \in \mathcal{L}^1(\mathcal{H})$ since $(1+D^2)^{-1/2} \in \mathcal{L}^P(\mathcal{H})$ by assumption and
- $(1+D^2)^{P/2} e^{-tD^2} = h(D) \in B(\mathcal{H})$, since $h: \mathbb{R}\to\mathbb{R},\ x \mapsto (1+x^2)^{P/2} e^{-tx^2}$ is bounded.

⌈Indeed, $\lim_{x\to\pm\infty} h(x) = 0$ and since

$$h'(x) = \frac{P}{2}(1+x^2)^{\frac{P}{2}-1} 2x e^{-tx^2} + (1+x^2)^{\frac{P}{2}} e^{-tx^2}(-2tx)$$

$$= (1+x^2)^{\frac{P}{2}-1} x e^{-tx^2}\left(\left(\frac{P}{2t}-1\right) - x^2\right),$$

we conclude that if $t < \frac{P}{2}$, then h has
- a local minimum at $x=0$ and
- global maxima at $x = \pm\sqrt{\frac{P}{2t}-1}$, namely
$$h\left(\pm\sqrt{\frac{P}{2t}-1}\right) = \left(\frac{P}{2t}\right)^{P/2} e^{-t\left(\frac{P}{2t}-1\right)} = \underbrace{\left(\frac{P}{2e}\right)^{P/2} e^{t}}_{<\left(\frac{P}{2e}\right)^{P/2} e^{P/2}} t^{-P/2},$$

and if $t \geq \frac{P}{2}$, then h has a global maximum at $x=0$, namely $h(0)=1$.⌋

Thus $e^{-tD^2} \in \mathcal{L}^1(\mathcal{H})$ and $\mathrm{Tr}(e^{-tD^2}) \leq C t^{-P/2}$ for $0<t<t_0$

Exercise 2:

We only have to check that

$$\|\cdot\| : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to [0,\infty), \quad T \mapsto \mathrm{Tr}_\omega |T|$$

satisfies the triangle inequality

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\| \quad \forall\, T_1, T_2 \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

This can be verified as follows:

$$\begin{aligned}
\|T_1 + T_2\| &= \mathrm{Tr}_\omega |T_1 + T_2| \\
&= \omega\Big(\big(\gamma_N(|T_1+T_2|)\big)_{N=1}^{\infty}\Big) \\
&\overset{(*)}{\leq} \omega\Big(\big(\gamma_N(|T_1|)\big)_{N=1}^{\infty}\Big) + \omega\Big(\big(\gamma_N(|T_2|)\big)_{N=1}^{\infty}\Big) \\
&= \mathrm{Tr}_\omega |T_1| + \mathrm{Tr}_\omega |T_2| \\
&= \|T_1\| + \|T_2\|,
\end{aligned}$$

where $(*)$ relies on that by Proposition 4.7 for each $N \in \mathbb{N}$

$$\begin{aligned}
\gamma_N(|T_1+T_2|) &= \gamma_N(T_1+T_2) \\
&\leq \gamma_N(T_1) + \gamma_N(T_2) \\
&= \gamma_N(|T_1|) + \gamma_N(|T_2|)
\end{aligned}$$

and that ω is positive and linear.

Remark: It is true (but not obvious) that for given T_1, T_2, we find isometries V_1, V_2 such that $|T_1+T_2| \leq V_1|T_1|V_1^* + V_2|T_2|V_2^*$; since Tr_ω is a hypertrace, this proves the assertion, too.