# Homology and Cohomology

## 1 Chain and cochain complexes

A chain complex \((C_\bullet, d_\bullet)\) is a family \((C_n)_{n \in \mathbb{Z}}\) of abelian groups \(C_n\) together with a family \(d_\bullet = (d_n)_{n \in \mathbb{Z}}\) of homomorphisms \(d_n : C_n \to C_{n-1}\), called boundary maps, which satisfy the condition \(d_n \circ d_{n+1} = 0\) for all \(n \in \mathbb{Z}\). Elements in \(Z_n(C_\bullet) := \ker d_n\) are called cycles and elements in \(B_n(C_\bullet) := \operatorname{ran} d_{n+1}\) are called boundaries; we define the \(n\)-the homology group by \(H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)\) for each \(n \in \mathbb{Z}\).

Chain complexes are often bounded from above in the sense that \(C_n = \{0\}\) for sufficiently large \(n\), say for \(n > n_0\); by an index shift, we may suppose that \(n_0 = 0\), i.e.,

\[
\cdots \xrightarrow{d_{k+2}} C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} \cdots \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \xrightarrow{d_0} \{0\}.
\]

Dually, a cochain complex \((C^\bullet, d^\bullet)\) is a family \((C^n)_{n \in \mathbb{Z}}\) of abelian groups \(C^n\) together with a family \(d^\bullet = (d^n)_{n \in \mathbb{Z}}\) of homomorphisms \(d^n : C^n \to C^{n+1}\), called differentials, which satisfy the condition \(d_{n+1} \circ d_n = 0\) for all \(n \in \mathbb{Z}\). Elements in \(Z^n(C^\bullet) := \ker d_n\) are called cocycles and elements in \(B^n(C^\bullet) := \operatorname{ran} d_{n-1}\) are called coboundaries; we define the \(n\)-the cohomology group by \(H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet)\) for each \(n \in \mathbb{Z}\).

Cochain complexes are often bounded from below. Analogous to the case of chain complexes, this means that we have \(C^n = \{0\}\) for sufficiently small \(n\); by an index shift, we may suppose that the cochain complex looks like

\[
\{0\} \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots.
\]

## 2 Hochschild homology and cohomology

Let \(A\) be a complex unital algebra and let \(M\) be an \(A\)-bimodule. The chain complex that underlies the Hochschild homology is obtained as follows. We put \(C_n(A, M) := M \otimes A^\otimes n\) for each \(n \geq 0\) and define the Hochschild boundary operator...
Let \( b_n : C_n(A, M) \to C_{n-1}(A, M) \) for \( n \geq 1 \) be defined by
\[
b_n(m \otimes a_1 \otimes \cdots \otimes a_n) := (ma_1) \otimes a_2 \otimes \cdots \otimes a_n
\]
\[
+ \sum_{k=1}^{n-1} (-1)^k m \otimes a_1 \otimes \cdots \otimes (a_ka_{k+1}) \otimes \cdots \otimes a_n
\]
\[
+ (-1)^n (a_nm) \otimes a_1 \otimes \cdots \otimes a_{n-1}
\]
and for \( n = 0 \) by \( b_0 = 0 \). We put \( Z_n(A, M) := \ker b_n \) and \( B_n(A, M) := \text{ran } b_{n+1} \); the Hochschild homology is defined as \( HH_n(A, M) = Z_n(A, M) / B_n(A, M) \) for each integer \( n \geq 0 \).

On the other hand, Hochschild cohomology is constructed as follows. We put \( C^n(A, M) := \text{hom}(A^\otimes n, M) \) for each \( n \geq 0 \); note that in particular \( C^0(A, M) = M \). The Hochschild differentials \( \delta^n : C^n(A, M) \to C^{n+1}(A, M) \) for \( n \geq 1 \) are given by
\[
(\delta^n \varphi)(a_1 \otimes \cdots \otimes a_{n+1}) := a_1 \varphi(a_2 \otimes \cdots \otimes a_{n+1})
\]
\[
+ \sum_{k=1}^{n} (-1)^k \varphi(a_1 \otimes \cdots \otimes (a_ka_{k+1}) \otimes \cdots \otimes a_n)
\]
\[
+ (-1)^{n+1} \varphi(a_1 \otimes \cdots \otimes a_n)a_{n+1}
\]
for each \( \varphi \in C^n(A, M) \) and \( a_1, \ldots, a_{n+1} \in A \). We put \( Z^n(A, M) := \ker \delta^n \) and \( B_n(A, M) := \text{ran } \delta^{n-1} \); the Hochschild cohomology is defined as \( HH^n(A, M) = Z^n(A, M) / B^n(A, M) \) for each integer \( n \geq 0 \).