UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Tobias Mai

Introduction to Noncommutative Differential Geometry

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Homology and Cohomology

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1 Chain and cochain complexes

A *chain complex* $(C_\bullet, d_\bullet)$ is a family $(C_n)_{n\in\mathbb{Z}}$ of abelian groups C_n together with a family $d_\bullet = (d_n)_{n\in\mathbb{Z}}$ of homomorphisms $d_n : C_n \to C_{n-1}$, called *boundary maps*, which satisfy the condition $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Elements in $Z_n(C_\bullet) := \ker d_n$ are called *cycles* and elements in $B_n(C_\bullet) := \operatorname{ran} d_{n+1}$ are called *boundaries*; we define the n-the homology group by $H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)$ for each $n \in \mathbb{Z}$.
Chain complexes are often *bounded from above* in the sense that $C_n = \{0\}$ for sufficiently large n, say for $n > n_0$; by an index shift, we may suppose that $n_0 = 0$, i.e.,

$$\ldots \xrightarrow{d_{k+2}} C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} \ldots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \{0\}.$$

Dually, a *cochain complex* $(C^\bullet, d^\bullet)$ is a family $(C^n)_{n\in\mathbb{Z}}$ of abelian groups C^n together with a family $d^\bullet = (d^n)_{n\in\mathbb{Z}}$ of homomorphisms $d^n : C^n \to C^{n+1}$, called *differentials*, which satisfy the condition $d_{n+1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Elements in $Z^n(C^\bullet) := \ker d_n$ are called *cocycles* and elements in $B^n(C^\bullet) := \operatorname{ran} d_{n-1}$ are called *coboundaries*; we define the n-the cohomology group by $H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet)$ for each $n \in \mathbb{Z}$.
Cochain complexes are often *bounded from below*. Analogous to the case of chain complexes, this means that we have $C^n = \{0\}$ for sufficiently small n; by an index shift, we may suppose that the cochain complex looks like

$$\{0\} \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \ldots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \ldots.$$

2 Hochschild homology and cohomology

Let A be a complex unital algebra and let M be an A-bimodule.
The chain complex that underlies the Hochschild homology is obtained as follows. We put $C_n(A, M) := M \otimes A^{\otimes n}$ for each $n \geq 0$ and define the Hochschild boundary operator

$b_n : C_n(A, M) \to C_{n-1}(A, M)$ for $n \geq 1$ by

$$\begin{aligned} b_n(m \otimes a_1 \otimes \cdots \otimes a_n) := (ma_1) &\otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{k=1}^{n-1} (-1)^k m \otimes a_1 \otimes \cdots \otimes (a_k a_{k+1}) \otimes \cdots \otimes a_n \\ &+ (-1)^n (a_n m) \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

and for $n = 0$ by $b_0 = 0$. We put $Z_n(A, M) := \ker b_n$ and $B_n(A, M) := \operatorname{ran} b_{n+1}$; the Hochschild homology is defined as $HH_n(A, M) = Z_n(A, M)/B_n(A, M)$ for each integer $n \geq 0$.

On the other hand, Hochschild cohomology is constructed as follows. We put $C^n(A, M) := \hom(A^{\otimes n}, M)$ for each $n \geq 0$; note that in particular $C^0(A, M) = M$. The Hochschild differentials $\delta^n : C^n(A, M) \to C^{n+1}(A, M)$ for $n \geq 1$ are given by

$$\begin{aligned} (\delta^n \varphi)(a_1 \otimes \cdots \otimes a_{n+1}) := a_1 &\varphi(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{k=1}^{n} (-1)^k \varphi(a_1 \otimes \cdots \otimes (a_k a_{k+1}) \otimes \cdots \otimes a_n) \\ &+ (-1)^{n+1} \varphi(a_1 \otimes \cdots \otimes a_n) a_{n+1} \end{aligned}$$

for each $\varphi \in C^n(A, M)$ and $a_1, \ldots, a_{n+1} \in A$. We put $Z^n(A, M) := \ker \delta^n$ and $B_n(A, M) := \operatorname{ran} \delta^{n-1}$; the Hochschild cohomology is defined as $HH^n(A, M) = Z^n(A, M)/B^n(A, M)$ for each integer $n \geq 0$.