Introduction to noncommutative differential geometry

held by Dr. Tobias Mai in Summer '19

UNIVERSITÄT DES SAARLANDES

Contents

Chapter 1.

Introduction

Several "classical theories" in mathematics can be extended to the noncommutative world. The appropriate framework is often obtained by the following recipe:

(i) Take a *classical space*, i.e., a set X endowed with some additional structure (e.g. a topological- or measure space, groups, manifolds, Lie groups, ...);

(ii) Consider a suitable algebra of functions on X (e.g., $C_0(X)$, $C(X)$, $L^\infty(X)$, $C^\infty(X)$, ...);

(iii) Transfer the additional structure of the space X to its associated (commutative) algebra of functions and provide an intrinsic characterisation of that structure;

(iv) Drop the assumption of commutativitiy.

Finding a good axiomatic description in (iii) that allows one to perform step (iv) is clearly the core problem and by no means straight foreward. The right choice confirms itself by a "reconstruction theorem", by which, in the commutative case, the underlying space can be "recovered" from that set of axioms. We list some prominent examples in Table 1.1.

The actual "noncommutative space" is mostly just a "virtual" object behind those algebras. The classical theories are thus rebuilt in an algebraic way, immitating the dual picture on their associated algebras of functions.

This philosophy underlies also the theory of noncommutative differential geometry that Alain Connes began to develop around the 80's. His motivation was to extend classical tools to

- spaces, that are "badly behaved" as point sets, but correspond naturally to (noncommutative) algebras (e.g., Penrose tilings, the space of leaves of a foliation, the phase space in quantum mechanics, ...)

- general noncommutative situations without an underlying space.

But even for classical situations that are purely commutative, this point of view gives new insights. Within noncommutative differential geometry, manifolds are studied by some spectral data. The following definition is at the heart of that approach.

Definition 1.1 (Alain Connes, 1994): A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where

- $\mathcal{A}$ is a unital complex *-algebra,
- $\mathcal{H}$ is a separable complex Hilbert space with a faithful *-representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$,
- $\mathcal{D}$ is a (possibly unbounded) selfadjoint linear operator on $\mathcal{H}$, say

$$\mathcal{D}\colon \mathcal{H} \supseteq \operatorname{dom} \mathcal{D} \longrightarrow \mathcal{H},$$

  with compact resolvents, i.e., $(\mathcal{D} - \lambda 1)^{-1} \in K(\mathcal{H})$ for all $\lambda \in \mathbb{C} - \sigma(\mathcal{D})$,

such that for all elements $a \in \mathcal{A}$ the following holds: $\pi(a) \operatorname{dom} \mathcal{D} \subseteq \operatorname{dom} \mathcal{D}$ and the commutator $[\mathcal{D}, \pi(a)] := \mathcal{D}\pi(a) - \pi(a)\mathcal{D}$ extends to an operator in $B(H)$.

Example 1.2: Consider on $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ the *arc length measure* m, i.e., the push-foreward of the Lebesgue measure on $\mathbb{R}$ via the map

$$\gamma\colon \mathbb{R} \longrightarrow \mathbb{T}, \qquad t \longmapsto \exp(\mathrm{i}t).$$

A function $f\colon \mathbb{T} \to \mathbb{C}$ is differentiable if and only if $f \circ \gamma\colon \mathbb{R} \to \mathbb{C}$ is so; its derivative $f'\colon \mathbb{T} \to \mathbb{C}$ is determined by $f'(\gamma(t))\gamma'(t) = (f \circ \gamma)'(t)$ for all $t \in \mathbb{R}$. Take now $\mathcal{H} = L^2(\mathbb{T}, m)$ and $\mathcal{A} = C^\infty(\mathbb{T})$ with the *-representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ given by

$$\pi(f) := M_f\colon L^2(\mathbb{T}, m) \longrightarrow L^2(\mathbb{T}, m), \qquad g \longmapsto fg.$$

Further, we consider the densely defined operator

$$\mathcal{D}_0\colon \mathcal{H} \supseteq \operatorname{dom} \mathcal{D}_0 \longrightarrow \mathcal{H}, \qquad g \longmapsto \frac{1}{\mathrm{i}} g'$$

on $\operatorname{dom} \mathcal{D}_0 := C^1(\mathbb{T})$, which is a symmetric operator. One can show that $\mathcal{D}_0$ is essentially self-adjoint; let $\mathcal{D}$ be its closure, which is thus selfadjoint. Then

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple. Indeed, if $\mathcal{F}\colon \mathcal{H} \to \ell^2(\mathbb{Z})$ is the Fourier transform, i.e.,

$$\mathcal{F}\colon L^2(\mathbb{T}, m) \longrightarrow \ell^2(\mathbb{Z}), \qquad f \longmapsto (\hat{f}_n)_{n\in\mathbb{Z}}, \quad \hat{f}_n := \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\zeta^{-n}\, dm(\zeta),$$

then $\mathcal{F}\mathcal{D}\mathcal{F}^{-1}$ is the multiplication by $(n)_{n\in\mathbb{Z}}$, hence we see that for all $\lambda \in \mathbb{C} - \mathbb{Z}$ it holds $(\mathcal{D} - \lambda)^{-1} \in K(\mathcal{H})$; moreover, for $f \in \mathcal{A}$ and $g \in C^1(\mathbb{T})$, it holds

$$[\mathcal{D}, \pi(f)]g = \mathcal{D}_0(fg) - f\mathcal{D}_0 g = \frac{1}{\mathrm{i}}\pi(f')g.$$

We will see, that more general manifolds $\mathcal{M}$ induce spectral triples in a similar way. Much of the structure of $\mathcal{M}$ can be recovered:

- $d(p, q) := \sup\{|f(p) - f(q)| \mid f \in \mathcal{A} : \|[\mathcal{D}, \pi(f)]\| \le 1\}$ is the geodesic distance between $p, q \in \mathcal{M}$,
- $\int_{\mathcal{M}} f\, d\mathrm{vol} = c(n)\mathrm{Tr}(f|\mathcal{D}|^{-n})$ for all $f \in \mathcal{A}$.

Exercises

Exercise 1.1: Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and let $V \in B(\mathcal{H})$ be any selfadjoint operator. Prove that $(\mathcal{A}, \mathcal{H}, \mathcal{D}_V)$ for the unbounded operator $\mathcal{D}_V$ given by $\mathcal{D}_V := \mathcal{D} + V$ with domain $\mathrm{dom}(\mathcal{D}_V) := \mathrm{dom}(\mathcal{D})$ is again a spectral triple.

classical space	noncommutative counterpart	reconstruction theory
locally compact (compact) Hausdorff topological space	(unital) C^*-algebras, noncommutative topology	Gelfand-Naimark theorem, FA I, Corollary 10.17
compact Hausdorff topological space with finite radon measure	von Neumann algebras, noncommutative measure theory	FA II, Theorem 8.15
compact topological group	compact quantum group, "noncommutative" group theory	Tannaka-Krein theorem
compact oriented smooth manifold	spectral triple (unbounded) K-cycle[1], noncommutative differential geometry	Connes' reconstruction theorem[2]

Table 1.1.: Examples for outputs of the noncommutiser

There are several important "noncommutative spaces" that generalise classical ones. Although finding the right axiomatic framework was in none of the cases straightforward, all these noncommutative counterparts have in common that they mimic algebras of suitable functions over the respective classical spaces. For instance, the definition of a C^*-algebra is modeled on $C(X)$, i.e., the space of all continuous functions $f\colon X \to \mathbb{C}$ on a compact Hausdorff topological space X: It forms a unital complex *-algebra (with operations defined pointwise and the involution given by $f \mapsto \overline{f}$), it becomes a Banach algebra when endowed with the maximum norm $\|f\| := \max_{x\in X} |f(x)|$, and all its elements f satisfy the crucial identity $\|f\overline{f}\| = \|f\|^2$.

[2]Connes, 1994

[2]2013

Chapter 2.

Spectral triples associated to manifolds

Spectral triples are (supposed to be) the right framework to extend classical differential geometry to the noncommutative world. It is however not clear offhand, how usual manifolds fit into that frame. In this chapter, we will see that indeed each compact oriented smooth manifold induces a commutative spectral triple in a natural fashion.

Definition 2.1 (Manifolds):

(i) An *n-dimensional topological manifold* is a Hausdorff topological space $\mathcal{M}$ which is *locally euclidean*, i.e., each point $x \in \mathcal{M}$ has an open neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

(ii) A *(local) chart* (U, φ) of $\mathcal{M}$ consists of an open subset $U \subseteq \mathcal{M}$ and a homeomorphism $\varphi\colon U \to \Omega = \varphi(U) \subseteq \mathbb{R}^n$.

(iii) A family $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$ of charts satisfying $M = \bigcup_{i \in I} U_i$ is called an *atlas of* $\mathcal{M}$. The homeomorphisms $\psi_{i,j}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ given by $\psi_{i,j} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)}$ are called *transition maps.*

(iv) An atlas $\mathcal{A}$ of $\mathcal{M}$ is called *smooth* if all its transition maps are smooth, i.e., C^∞. A chart (U, φ) is said to be *smooth with respect to a smooth atlas* $\mathcal{A}$, if $\mathcal{A} \cup \{(U, \varphi)\}$ is again a smooth atlas. A smooth atlas $\mathcal{A}$ is called *maximal*, if every chart (U, φ) that is smooth with respect to $\mathcal{A}$ already belongs to $\mathcal{A}$. Every smooth atlas $\mathcal{A}$ induces a maximal one by

$$\mathcal{A}_{\max} := \{(U, \varphi) \mid (U, \varphi) \text{ is a chart smooth with respect to } \mathcal{A}\}.$$

(v) An *n-dimensional smooth manifold* is an n-dimensional topological manifold $\mathcal{M}$ with a maximal smooth atlas $\mathcal{A}$.

Definition 2.2 (Tangent space): Let $\mathcal{M}$ be an n-dimensional smooth manifold. We fix $x_0 \in \mathcal{M}$.

(i) A function $f\colon V \to \mathbb{R}$ on an open subset $V \subseteq \mathcal{M}$ is said to be *smooth*, if $f \circ \varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U \cap V) \to \mathbb{R}$ is smooth for every smooth chart (U, φ).

(ii) On the set of all pairs (V, f) consisting of an open neighbourhood V of x_0 and a smooth function $f\colon V \to \mathbb{R}$, we introduce an equivalence relation $\sim$ by

$$(V_1, f_1) \sim (V_2, f_2) :\Longleftrightarrow \exists\, V \subseteq V_1 \cap V_2 \text{ open}, x_0 \in V : f_1|_V = f_2|_V.$$

The equivalence class of (V, f), denoted by $[f]_{x_0}$, is called the *germ of f at x_0*. We denote by $C^\infty_{x_0}(\mathcal{M})$ the $\mathbb{R}$-algebra of germs at x_0.

(iii) The $\mathbb{R}$-vector space $T_{x_0}\mathcal{M}$ of all linear maps $\delta\colon C^\infty_{x_0}(\mathcal{M}) \to \mathbb{R}$ satisfying the product rule

$$\delta([f]_{x_0} \cdot [g]_{x_0}) = \delta([f]_{x_0}) \cdot g(x_0) + f(x_0) \cdot \delta([g]_{x_0}) \quad \forall\, [f]_{x_0}, [g]_{x_0} \in C^\infty_{x_0}(\mathcal{M})$$

is called the *tangent space to $\mathcal{M}$ at x_0*.

Remark 2.3: In the situation of Definition 2.2, let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ be a smooth path, i.e., γ is continuous and $\varphi \circ \gamma|_{\gamma^{-1}(U)}\colon \gamma^{-1}(U) \to \mathbb{R}^n$ is smooth for every smooth chart (U, φ), such that $\gamma(0) = x_0$. We call $\gamma'(0) \in T_{x_0}\mathcal{M}$ given by

$$\gamma'(0)([f]_{x_0}) := (f \circ \gamma)'(0)$$

for all germs $[f]_{x_0} \in C^\infty_{x_0}(\mathcal{M})$, the *velocity vector of γ at x_0*.

For every $\delta \in T_{x_0}\mathcal{M}$, there exists a smooth path $\gamma\colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = x_0$ and $\gamma'(0) = \delta$.

Next, we “glue” the tangent spaces, yielding the so-called *tangent bundle*.

Definition 2.4 (Topological vector bundle): Let X be a Hausdorff topological space.

(i) An *n-dimensional (real / complex) vector bundle over X* is given by a topological space E and a continuous map $\pi\colon E \to X$ such that the following conditions are satisfied:

- The *fibre* $E_x := \pi^{-1}(\{x\})$ is a real / complex vector space of dimension n for each $x \in X$.

- For each $x_0 \in X$, there is an open neighbourhood U of x_0 and a homemorphism $\tau\colon \pi^{-1}(U) \to U \times \mathbb{K}^n$[1] such that $\pi|_{\pi^{-1}(U)} = \mathrm{pr}_U \circ \tau$, where $\mathrm{pr}_U\colon U \times \mathbb{K}^n \to U$ is the projection onto the first component, i.e., $\mathrm{pr}_U(x,v) = x$, and $\tau|_{E_x}\colon E_x \to \{x\} \times \mathbb{K}^n \cong \mathbb{K}^n$ is a vector space isomorphism for all points $x \in U$. We call (U,τ) a *bundle chart* (or a *local trivialisation*).

(ii) A family $\mathcal{A} = \{(U_i,\tau_i) \mid i \in I\}$ of bundle charts (or local trivialisations) satisfying $X = \bigcup_{i\in I} U_i$ is called a *bundle atlas.* The transition maps

$$\sigma_{i,j}\colon (U_i \cap U_j) \times \mathbb{K}^n \longrightarrow (U_i \cap U_j) \times \mathbb{K}^n$$

given by $\sigma_{i,j} := \tau_j \circ \tau_i^{-1}|_{(U_i\cap U_j)\times\mathbb{K}^n}$ satisfying

$$\begin{tikzcd}
 & \pi^{-1}(U_i \cap U_j) \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & \\
(U_i \cap U_j) \times \mathbb{K}^n \arrow[rr, "\sigma_{i,j}", "(x,v)\mapsto\sigma_{i,j}(x,v)=(x,S_{i,j}(x)v)"'] & & (U_i \cap U_j) \times \mathbb{K}^n
\end{tikzcd}$$

for a continuous map $S_{i,j}\colon U_i \cap U_j \to \mathrm{Gl}_n(\mathbb{K})$ called the *transition maps.*

Definition 2.5 (Smooth vector bundle): Let $\mathcal{M}$ be a smooth manifold. An *n-dimensional smooth vector bundle over $\mathcal{M}$* is an n-dimensional topological vector bundle over $\mathcal{M}$, for which all transition maps are smooth.

Definition 2.6 (Tangent bundle): Let $\mathcal{M}$ be an n-dimensional smooth manifold with maximal smooth atlas $\mathcal{A} = \{(U_i,\varphi_i) \mid i \in I\}$. We put $T\mathcal{M} := \coprod_{x\in\mathcal{M}} T_x\mathcal{M}$ and define $\pi\colon T\mathcal{M} \to \mathcal{M}$ by $\pi(\delta) = x$ if $\delta \in T_x\mathcal{M}$. We define the local trivialisation by

$$\tau_i\colon \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n, \qquad (x,\delta) \longmapsto (x, \Theta_{i,x}^{-1}(\delta))$$

with the isomorphism $\Theta_{i,x}\colon \mathbb{R}^n \to T_x\mathcal{M}$ given by

$$\Theta_{i,x}(v)([f]_x) := \sum_{j=1}^{n} v^j \partial_j (f \circ \varphi_i^{-1})(\varphi_i(x))$$

for $v = (v^1,\dots,v^n) \in \mathbb{R}^n$, $[f]_x \in C_x^\infty(\mathcal{M})$. We endow $T\mathcal{M}$ with the topology defined by the following requirement:

$$W \subseteq T\mathcal{M} \text{ open} :\Longleftrightarrow \forall\, i \in I : \tau_i(U_i \cap W) \subseteq U_i \times \mathbb{R}^n \text{ is open.}$$

Then $T\mathcal{M}$ is an n-dimensional smooth vector bundle over $\mathcal{M}$, called the *tangent bundle of $\mathcal{M}$.*

[1] Here, $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Definition 2.7 (Smooth sections): Let $\pi\colon E \to \mathcal{M}$ be a smooth vector bundle over a smooth manifold $\mathcal{M}$ and let $V \subseteq M$ be open. A map $s\colon V \to E$ is called a *smooth section* if

(i) $\pi \circ s = \mathrm{id}_V$,

(ii) For each local trivialisation $\tau\colon \pi^{-1}(U) \to U \times \mathbb{K}^n$, we have a smooth composition $\tau \circ s|_{U\cap V}\colon U \cap V \to (U \cap V) \times \mathbb{K}^n$. In fact, $(\tau \circ s)(x) = (x, f(x))$ for all $x \in U \cap V$ with $f\colon U \cap V \to \mathbb{K}$ being a smooth function.

We write $\Gamma^\infty(V, E)$ for the set of all smooth sections $s\colon V \to E$, which is a $\mathbb{K}$-vector space.

Definition 2.8 (Vector fields): Let $\mathcal{M}$ be a smooth manifold. A *vector field on* $\mathcal{M}$ is a smooth section of the tangent bundle $T\mathcal{M}$. We write

$$\mathfrak{X}(V) := \Gamma^\infty(V, T\mathcal{M})$$

for the vector fields on the open subset $V \subseteq \mathcal{M}$.

Theorem 2.9: *Let $\mathcal{M}$ be a smooth manifold of dimension n and let $V \subseteq \mathcal{M}$ be open. Then the map*

$$\Phi\colon \mathfrak{X}(V) \longrightarrow \mathrm{der}(C^\infty(V)), \qquad (\Phi(X)f)(x) := X(x)([f]_x)$$

is an isomorphism of real vector spaces.

Note that $\mathrm{der}(C^\infty(V))$ denotes the space of *derivations* on $C^\infty(V)$, i.e., linear maps $D\colon C^\infty(V) \to C^\infty(V)$ that satisfy the product rule for all $f, g \in C^\infty(V)$:

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g).$$

Proof: (1) Take $X \in \mathfrak{X}(V)$ and $f \in C^\infty(V)$. Then the map

$$h\colon x \longrightarrow \mathbb{R}, \qquad x \longmapsto X(x)([f]_x)$$

is smooth.

To see this, consider the bundle atlas $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ that we introduced in Definition 2.6. It suffices to show that $h|_{U_i \cap V}$ is smooth for all $i \in I$. By Definition 2.7 (ii), we find a smooth function $g = (g_1, \dots, g_n)\colon U_i \cap V \to \mathbb{R}^n$ such that $\tau_i(X(x)) = (x, g(x))$ for all $x \in U_i \cap V$. Thus

$$X(x) = \tau_i^{-1}(x, g(x)) = (x, \Theta_{i,x}(g(x)))$$

and $h(x) = X(x)([f]_x) = \Theta_{i,x}(g(x))([f]_x) = \sum_{j=1}^n g_j(x)(\partial_j(f \circ \varphi_i^{-1})(\varphi_i(x))$ for all $x \in U_i \cap V$, which shows that h is smooth on $U_i \cap V$.

(2) Take $X \in \mathfrak{X}(V)$. Because of the preceeding consideration, we have a well-defined map

$$\Phi(X)\colon C^\infty(V) \longrightarrow C^\infty(V), \qquad (\Phi(X)f)(x) = X(x)([f]_x).$$

This $\Phi(X)$ is now in fact a deriavation.

To see this, let $f, g \in C^\infty(V)$ be given. Then, for all $x \in V$, we have

$$\begin{aligned}(\Phi(X)(f \cdot g))(x) = X(x)([f]_x \cdot [g]_x) &= X(x)([f]_x)g(x) + f(x)X(x)([g]_x) \\ &= (\Phi(X)(f)) \cdot g + f \cdot (\Phi(X)g))(x),\end{aligned}$$

as desired.

(3) By the above steps, we now have a well-defined map

$$\Phi\colon \mathfrak{X}(V) \longrightarrow \operatorname{der}(C^\infty(V)), \qquad X \longmapsto \Phi(X),$$

which is clearly linear.

(4) For the next steps, we need a technical result. Let U be an open neighbourhood of a point $x_0 \in \mathcal{M}$. There exists a smooth function $\rho\colon \mathcal{M} \to [0,1]$ with compact support

$$\operatorname{supp}(\rho) = \operatorname{cl}(\{x \in \mathcal{M} \mid \rho(x) \neq 0\}) \subseteq U$$

which is identically 1 in an open neighbourhood of x_0. We call ρ a *bump function for* (U, x_0) (there are such bump functions, take one in $\mathbb{R}^n$ and transport it to $\mathcal{M}$ via a chart).

(5) Take $D \in \operatorname{der}(C^\infty(V))$ and $x_0 \in V$. We define the map

$$D|_{x_0}\colon C^\infty(V) \longrightarrow \mathbb{R}, \qquad f \longmapsto (D(f))(x_0).$$

If $f_1, f_2 \in C^\infty(V)$ satisfy $f_1|_U = f_2|_U$ for an open set $U \subseteq V$ with $x_0 \in U$, then $D|_{x_0}(f_1) = D|_{x_0}(f_2)$.

To see this, take a bump function ρ for (U, x_0). Since $\rho \cdot (f_1 - f_2) \equiv 0$ we get that

$$\begin{aligned}0 = D|_{x_0}(\rho \cdot (f_1 - f_2)) &= D|_{x_0}(\rho)(f_1 - f_2)(x_0) + \rho(x_0) \cdot D|_{x_0}(f_1 - f_2) \\ &= D|_{x_0}(f_1) - D|_{x_0}(f_2)\end{aligned}$$

(as f_1 and f_2 agree on U and $\rho(x_0) = 1$) and thus $D|_{x_0}(f_1) = D|_{x_0}(f_2)$, as desired.

(6) We thus get a well-defined map

$$(\Psi(D))(x_0)\colon C^\infty_{x_0}(V) \longrightarrow \mathbb{R}, \qquad [f]_{x_0} \longmapsto D|_{x_0}(\rho \cdot f|_V)$$

where, for a (U, f) representing $[f]_{x_0}$, ρ is any bump function for (U, x_0) and $\rho f \in C^\infty(\mathcal{M})$ is defined (in fact well-defined) by

$$(\rho f)(x) = \begin{cases} 0, & \text{if } x \notin \operatorname{supp}(\rho), \\ \rho(x) f(x), & \text{if } x \in U. \end{cases}$$

Clearly, $(\Psi(D))(x_0) \in T_{x_0}\mathcal{M}$.

(7) If we now take a derivation $D \in \operatorname{der}(C^\infty(V))$, then the induced map

$$\Psi(D)\colon V \longrightarrow T\mathcal{M}, \qquad x \longmapsto (\Psi(D))(x)$$

belongs to $\mathfrak{X}(V)$.

(8) Finally, we get a linear map

$$\Psi\colon \operatorname{der}(C^\infty(V)) \longrightarrow \mathfrak{X}(V),$$

which satisfies $\Phi \circ \Psi = \mathrm{id}_{\operatorname{der}(C^\infty(V))}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{X}(V)}$. □

Remark 2.10: Like vector spaces underly linear algebra, vector bundles underly what can be seen as "parametrised" linear algebra. Indeed, various constructions for vector spaces can be generalised to that setting.

Let X be a Hausdorff topological space and let E and F be vector bundles over X (both real or complex) of dimension n and m, respectively.

(i) The *Whitney sum* (or *direct sum*) $E \oplus F$ is the vector bundle of dimension $n + m$ with $(E \oplus F)_x = E_x \oplus F_x$ for all $x \in X$.

(ii) The *tensor product bundle* $E \otimes F$ is the vector bundle of dimension $n \cdot m$ with $(E \otimes F)_x = E_x \otimes F_x$ for all $x \in X$.

(iii) The *homomorphisms bundle* $\hom(E, F)$ is the vector bundle of dimension $n \cdot m$ with $\hom(E, F)_x = \hom(E_x, F_x)$ for all $x \in X$.

Analogously, the *dual bundle* E^* (see assignment 2A, exercise 1 (i)), the *exterior product* $\bigwedge^p E$, the vector bundle $\operatorname{mult}^p(E)$ of all p-multilinear maps and the vector bundles $\operatorname{sym}^p(E)$ and $\operatorname{alt}^p(E)$ of all symmetric– respectively alternating p-multilinear maps can be defined.

Definition 2.11 (Riemannian metric): Let $\mathcal{M}$ be a smooth manifold of dimension n. A *Riemannian metric on* $\mathcal{M}$ is a smooth section g of the vector bundle $\mathrm{sym}^2(T\mathcal{M})$ such that

$$g_x \colon T_x\mathcal{M} \times T_x\mathcal{M} \longrightarrow \mathbb{R}$$

is an inner product on $T_x\mathcal{M}$ for all $x \in X$.

Remark 2.12: (i) A topological space X is said to be *paracompact* if every open cover $(U_i)_{i\in I}$ of X has an *open refinement* $(V_j)_{j\in J}$ (i.e., $(V_j)_{j\in J}$ is an open cover of X and for all $j \in J$ there is $i \in I$ such that $V_j \subseteq U_i$) that is *locally finite* (i.e., every $x \in X$ has a neighbourhood V such that only finitely many V_j $(j \in J)$ have non-trivial intersection with V).

(ii) Let X be a topological space. A *parition of unity on* X is a family $(\rho_i)_{i\in I}$ of continuous functions $\rho_i \colon X \to [0,1]$ such that

- Each $x \in X$ has an open neighbourhood V such that only finitely many ρ_i satisfy $\rho_i|_V \not\equiv 0$,
- For all $x \in X$ we have $\sum_{i\in I} \rho_i(x) = 1$.

We say that $(\rho_i)_{i\in I}$ is *subordinate to an open cover* $(U_i)_{i\in I}$ *of* X, if $\mathrm{supp}(\rho_i) \subseteq U_i$ for all $i \in I$.

(iii) On a paracompact space X, each open cover $(U_i)_{i\in I}$ of X has a subordinate partition of unity $(\rho_i)_{i\in I}$. If $X = \mathcal{M}$ is a paracompact smooth manifold, then each ρ_i can be chosen to be smooth.

Theorem 2.13: *Let $\mathcal{M}$ be a smooth manifold of dimension n. Suppose that $\mathcal{M}$ is paracompact. Then there is a Riemannian metric on $\mathcal{M}$.*

Proof: Take a bundle atlas $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ of $T\mathcal{M}$ and let $(\rho_i)_{i\in I}$ be a smooth partition of unity subordinate to $(U_i)_{i\in I}$. We obtain a Riemannian metric by

$$g_x(\delta_1, \delta_2) := \sum_{i\in I} \rho_i(x)\langle \Theta_{i,x}^{-1}(\delta_1), \Theta_{i,x}^{-1}(\delta_2)\rangle$$

for all $x \in \mathcal{M}$ and $\delta_1, \delta_2 \in T_x\mathcal{M}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. □

Definition 2.14 (Differential forms): Let $\mathcal{M}$ be a smooth manifold of dimension n.

(i) The dual bundle $T^*\mathcal{M}$ to the tangent bundle $T\mathcal{M}$ is called the *cotangent bundle.*

(ii) A *smooth differential form of degree* p (briefly p-form in the following) is a smooth section of $\bigwedge^p T^*\mathcal{M}$. We put $\Omega^p(\mathcal{M}) := \Gamma^\infty(\mathcal{M}, \bigwedge^p M)$. We call $\Omega^\bullet(\mathcal{M}) := \bigoplus_{p\geq 0} \Omega^p(\mathcal{M})$ the *exterior algebra*.

(iii) The *exterior derivative* is the unique family $(d^p)_{p\geq 0}$ of $\mathbb{R}$-linear maps $d^p\colon \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ satisfiying

- For all $f \in \Omega^0(\mathcal{M}) = C^\infty(M)$ and all $x \in \mathcal{M}$

$$(d^0 f)(x)\colon T_x\mathcal{M} \longrightarrow \mathbb{R}, \qquad \delta \longmapsto \delta([f]_x);$$

- For all $p \geq 0$ it holds $d^{p+1} \circ d^p = 0$,
- For all $\omega \in \Omega^p(\mathcal{M}), \eta \in \Omega^q(\mathcal{M})$ it holds

$$d^{p+q}(\omega \wedge \eta) = d^p(\omega) \wedge \eta + (-1)^p \omega \wedge d^q(\eta)$$

where $\wedge$ on $\Omega^\bullet(\mathcal{M})$ is defined pointwise, i.e., $(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x)$ in $\bigwedge^{p+q} T_x^*\mathcal{M}$.

(iv) Let $T^*\mathcal{M}_\mathbb{C}$ be the *complexification of* $T^*\mathcal{M}$, i.e., $T^*\mathcal{M}_\mathbb{C} = T^*\mathcal{M} \otimes (\mathcal{M} \times \mathbb{C})$, where $\mathcal{M} \times \mathbb{C}$ is the real *trivial bundle* (i.e., $\pi\colon \mathcal{M} \times \mathbb{C} \to \mathcal{M}$, $(x, \lambda) \mapsto x$) of dimension 2 (note that $\mathbb{C} \cong \mathbb{R}^2$). We put $\Omega^p_\mathbb{C}(\mathcal{M}) := \Gamma^\infty(\mathcal{M}, \bigwedge^p_\mathbb{C} T^*\mathcal{M}_\mathbb{C})$.

Definition 2.15 (Orientation perserving): Let U, V be open subsets of $\mathbb{R}^n$. A diffeomorphism $f = (f^1, \ldots, f^n)\colon U \to V$ is said to be *orientation preserving* if for all $x \in U$ it holds

$$\det([\partial_j f^i(x)]_{1\leq i,j\leq n}) > 0.$$

Definition 2.16 (Orientation of smooth manifolds): Let $\mathcal{M}$ be a smooth manifold.

(i) A smooth atlas is called *oriented* if all its transition maps are orientation preserving.

(ii) We say $\mathcal{M}$ is *orientable* if it admits an oriented smooth atlas.

(iii) An orientation on $\mathcal{M}$ is a maximal oriented smooth atlas.

Theorem 2.17 (Integration of smooth functions): *Let $\mathcal{M}$ be an oriented n-dimensional smooth manfold which is paracompact. Let g be a Riemannian metric on $\mathcal{M}$. Then there exists a unique linear map*

$$\int_\mathcal{M}\colon C_c^\infty(\mathcal{M}) \longrightarrow \mathbb{R}, \qquad f \longmapsto \int_\mathcal{M} f$$

on the subspace $C_c^\infty(\mathcal{M}) \subseteq C^\infty(\mathcal{M})$ *of compactly supported functions such that the following condition is satisfied: For every local chart* (U,φ) *in the maximal oriented smooth atlas* $\mathcal{A}$ *of* $\mathcal{M}$ *and for each* $f \in C_c^\infty(\mathcal{M})$ *with* $\mathrm{supp}(f) \subseteq U$ *we have that*

$$\int_{\mathcal{M}} f = \int_{\varphi(U)} (f \circ \varphi^{-1})(\det[g_{k,l}]_{1\leq k,l\leq n})^{1/2}\, d\lambda^n\,. \tag{2.1}$$

Here, λ^n *is the Lebesgue measure on* $\mathbb{R}^n$ *and the functions* $g_{k,l} \in C^\infty(\varphi(U))$ *are for each* $x \in U$ *determined by*

$$g_{k,l}(\varphi(x)) := g_x((d\varphi)(x)^{-1}(\partial_k|_{\varphi(x)}), (d\varphi)(x)^{-1}(\partial_l|_{\varphi(x)})$$

Note that $\{\partial_k|_{\varphi(x)} \mid 1 \leq k \leq n\}$ *is the basis of* $T_{\varphi(x)}\mathbb{R}^n$ *introduced in Exercise 2.1 (ii) and* $(d\varphi)(x)\colon T_x\mathcal{M} \to T_{\varphi(x)}\mathbb{R}^n$ *is the* differential of φ at x, *which is defined for* $\delta \in T_x\mathcal{M}$ *and* $[f]_{\varphi(x)} \in C^\infty_{\varphi(x)}(\mathbb{R}^n)$ *by*

$$((d\varphi)(x)\delta)([f]_{\varphi(x)}) = \delta([f\circ\varphi]_x).$$

In fact, $(d\varphi)(x)$ *is bijective since* φ *is bijective.*

Remark 2.18: (i) The assumption that $\mathcal{M}$ is oriented guarantees that the right-hand side of Eq. (2.1) is well-defined, i.e., independent of the particular choice of the chart (U,φ).

Using a partition of unity subordinate to the family $(U_i)_{i\in I}$ for a maximal oriented smooth atlas $\mathcal{A} = \{(U_i,\varphi_i) \mid i \in I\}$, say $(\rho_i)_{i\in I}$, one can then define for general $f \in C_c^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} f := \sum_{i\in I} \int_{\mathcal{M}} (\rho_i f).$$

Since $\mathrm{supp}(\rho_i f) \subseteq U_i$ for each $i \in I$.

(ii) Theorem 2.17 merges actually two different concepts, namely on the one hand the integration of compactly supported n-forms, i.e.,

$$\int_{\mathcal{M}} : \Omega_c^n(\mathcal{M}) \longrightarrow \mathbb{R}, \qquad \omega \longmapsto \int_{\mathcal{M}} \omega$$

which requires only that $\mathcal{M}$ is oriented and on the other hand the volume form $d\mathrm{vol} \in \Omega^n(\mathcal{M})$ of an oriented Riemannian manifold $(\mathcal{M},g)$; in general, $\omega \in \Omega^n(\mathcal{M})$ is called a *volume-form* if ω vanishes nowhere, and a paracompact smooth manifold $\mathcal{M}$ is orientable if and only if a volume form exists; in fact, fixing an equivalence class of volume forms specifies an orientation and vice versa;

$d\text{vol}$ is chosen such that $d\text{vol}(x)$, for each $x \in \mathcal{M}$, is normalised with respect to the inner product on $\bigwedge^n T_x^*\mathcal{M}$ induced by g, i.e., $\langle d\text{vol}(x), d\text{vol}(x)\rangle_{\bigwedge^n T_x^*\mathcal{M}} = 1$.

One can show that for all $f \in C_c^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} f = \int_{\mathcal{M}} f\, d\text{vol},$$

where $f\, d\text{vol} \in \Omega_c^n(\mathcal{M})$.

(iii) Let V be a finite dimensional real vector space and let $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$ be an inner product on V. We thus have an isomorphism $\Phi \colon V \to V^*, x \mapsto \langle \cdot, x\rangle$ which allows us to define an inner product $\langle \cdot, \cdot \rangle_{V^*} \colon V^* \times V^* \to \mathbb{R}$ for $\varphi, \psi \in V^*$ by

$$\langle \varphi, \psi \rangle_{V^*} := \langle \Phi^{-1}(\varphi), \Phi^{-1}(\psi)\rangle.$$

For every $p \in \mathbb{N}$, we extend the latter to an inner product

$$\langle \cdot, \cdot \rangle_{\bigwedge^p V^*} \colon \bigwedge^p V^* \times \bigwedge^p V^* \longrightarrow \mathbb{R}$$

b for $\varphi_1 \wedge \cdots \wedge \varphi_p,\ \psi_1 \wedge \ldots \psi_p \in \bigwedge^p V^*$ by

$$\langle \varphi_1 \wedge \cdots \wedge \varphi_p, \psi_1 \wedge \cdots \wedge \psi_p \rangle_{\bigwedge^p V^*} := \det([\langle \varphi_k, \psi_l \rangle_{V^*}]_{1 \leq k,l \leq n}).$$

When applied to each fibre of $T\mathcal{M}$ for an oriented paracompact smooth manifold $\mathcal{M}$ with respect to the inner product induced by a Riemannian metric g on $\mathcal{M}$, we get for each $p \geq 0$ an inner product $\langle \cdot, \cdot \rangle_{\Omega_c^p(\mathcal{M})} \colon \Omega_c^p(\mathcal{M}) \times \Omega_c^p(\mathcal{M}) \to \mathbb{R}$ for $\omega, \eta \in \Omega_c^p(\mathcal{M})$ by

$$\langle \omega, \eta \rangle_{\Omega_c^p(\mathcal{M})} := \int_{\mathcal{M}} \langle \omega(x), \eta(x) \rangle_{\bigwedge^p T_x^*\mathcal{M}}\, d\text{vol}(x)\,.$$

In the case $p = n$, $\langle \cdot, \cdot \rangle_{\bigwedge^p T_x^*(\mathcal{M})}$ was used in (iii). The latter extend naturally to inner products

$$\langle \cdot, \cdot \rangle_{\Omega_{\mathbb{C},c}^p(\mathcal{M})} \colon \Omega_{\mathbb{C},c}^p(\mathcal{M}) \times \Omega_{\mathbb{C},c}^p(\mathcal{M}) \longrightarrow \mathbb{C}.$$

Theorem 2.19 (Hodge-de Rham triple): *Let $\mathcal{M}$ be an oriented compact smooth manifold of dimension n with Riemannian metric g. Consider*

(i) *the unital complex *-algebra $\mathcal{A} := C^\infty(\mathcal{M}, \mathbb{C})$,*

(ii) *the separable complex Hilbert space* $\mathcal{H} := L^2(\bigwedge_{\mathbb{C}}^{\bullet} T^*\mathcal{M}, g)$*, which is obtained as the completion of the complex exterior algebra* $\Omega_{\mathbb{C}}^{\bullet}(\mathcal{M}) := \bigoplus_{p\geq 0} \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$ *with respect to the inner product given by*

$$\langle (\omega_0, \dots, \omega_n), (\eta_0, \dots, \eta_n) \rangle := \sum_{p=0}^{n} \langle \omega_p, \eta_p \rangle_{\Omega_{\mathbb{C}}^p(\mathcal{M})}$$

and the **-representation* $\pi \colon \mathcal{A} \to B(\mathcal{H})$ *given by multiplication, i.e.,* $\pi(f)(\omega) := f\omega$ *for every* $f \in \mathcal{A}$ *and* $\omega \in \mathcal{H}$*,*

(iii) *the unbounded operator* $\mathcal{D}_0 := d + d^*$*, where* d^* *is the adjoint of the densely defined operator*

$$d \colon \mathcal{H} \supseteq \operatorname{dom} d \longrightarrow \mathcal{H}, \qquad \omega \longmapsto d\mathrm{Re}(\omega) + \mathrm{i} d\mathrm{Im}(\omega)$$

with domain $\operatorname{dom} d := \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$*.*

Then $\mathcal{D}_0$ *is essentially self-adjoint; let* $\mathcal{D}$ *be its closure, which we call the* Hodge-de Rham operator*. The* Hodge-de Rham triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a commutative spectral triple in the sense of Definition 1.1. We call* $\Delta := \mathcal{D}^2$ *the* Hodge Laplacian*.*

Definition 2.20: Let V be a $\mathbb{K}$ vector space with an inner product $\langle \cdot, \cdot \rangle$. Put $\bigwedge_{\mathbb{K}}^{\bullet} V := \bigoplus_{p\geq 0} \bigwedge_{\mathbb{K}}^{p} V$. Then

$$\llcorner \colon V \times V \longrightarrow \bigwedge_{\mathbb{K}}^{\bullet} V, \qquad v \llcorner (v_1 \wedge \cdots \wedge v_p) := \sum_{k=1}^{p} (-1)^{k+1} \langle v_k, v \rangle v_1 \wedge \cdots \wedge \widehat{v}_k \wedge \cdots \wedge v_p.$$

Remark 2.21: The proof of Theorem 2.19 relies mostly on techniques that are (not yet) at our disposal. We can understand, however, how commutators $[\mathcal{D}, \phi(f)]$ for $f \in \mathcal{A}$ look on $\mathcal{H}$. They are given by the *Clifford multiplication* with df from the left, i.e., for all $\omega \in \Omega_{\mathbb{C}}^{\bullet}$ we have

$$[\mathcal{D}, \pi(f)]\omega = df \bullet w. \tag{2.2}$$

The Clifford multiplication is defined on fibres as follows: On the exterior algebra $\bigwedge_{\mathbb{C}}^{\bullet} V_{\mathbb{C}} := \bigoplus_{p\geq 0} \bigwedge_{\mathbb{C}}^{p} V_{\mathbb{C}}$ for the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ of a finite dimensional real Hilbert space $(V, \langle \cdot, \cdot \rangle)$, we define

$$\llcorner \colon V_{\mathbb{C}} \times \bigwedge_{\mathbb{C}}^{\bullet} V_{\mathbb{C}} \longrightarrow \bigwedge_{\mathbb{C}}^{\bullet} V_{\mathbb{C}}$$

by

$$v \llcorner (v_1 \wedge \cdots \wedge v_p) := \sum_{k=0}^{p} (-1)^{k+1} \langle v_k, \overline{v} \rangle_{\mathbb{C}} v_1 \wedge \cdots \wedge \widehat{v}_k \wedge \cdots \wedge v_p.$$

Note that, for $v = u \otimes \lambda \in V_{\mathbb{C}}$, we put $\overline{v} := u \otimes \overline{\lambda}$; the inner product $\langle \cdot, \cdot \rangle \colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is defined by

$$\langle u_1 \otimes \lambda_1, u_2 \otimes \lambda_2 \rangle := \langle u_1, u_2 \rangle \lambda_1 \overline{\lambda}_1.$$

Then $v \bullet \omega := v \wedge \omega - v \llcorner \omega$ for all $v \in V_{\mathbb{C}}$ and $\omega \in \bigwedge_{\mathbb{C}}^{\bullet}$.

(1) For all $f \in \mathcal{A}, \omega \in \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$ it holds

$$d^*(f\omega) = f d^* \omega - df \llcorner \omega.$$

To see this, we take $\eta \in \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$ and compute with respect to the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})}$ that

$$\begin{aligned} \langle d\eta, f\omega \rangle &= \langle \overline{f} d\eta, \omega \rangle \\ &= \langle d(\overline{f}\eta), \omega \rangle - \langle d\overline{f} \wedge \eta, \omega \rangle \\ &= \langle \overline{f}\eta, d^*\omega \rangle - \langle \eta, df \llcorner \omega \rangle = \langle \eta, f d^*\omega - df \llcorner \omega \rangle \end{aligned}$$

from which the assertion follows.

(2) For all $f \in \mathcal{A}$, $\omega \in \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$ it holds

$$[\mathcal{D}, \pi(f)] = df \bullet \omega$$

To see this, consider the following computation:

$$\begin{aligned} [\mathcal{D}, \pi(f)] &= [d, \pi(f)]\omega + [d^*, \pi(f)]\omega \\ &= d(f\omega) - f d\omega + d^*(f\omega) - f d^*\omega = df \wedge \omega - df \llcorner \omega = df \bullet \omega. \end{aligned}$$

That $[\mathcal{D}, \pi(f)]$ extends to bounded linear operator on $\mathcal{H}$, will be discussed later.

Further, we note that $\Omega_{\mathbb{C}}^{\bullet}(\mathcal{M}) \subseteq \operatorname{dom} d^*$, which justifies that $\mathcal{D}_0 = d + d^*$ is densely defined with $\operatorname{dom} \mathcal{D}_0 = \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$. This can be shown with the help of the *Hodge star operator* $* \colon \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M}) \to \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$ which associates to each $\omega \in \Omega_{\mathbb{C}}^{k}(\mathcal{M})$ the unique $*\omega \in \Omega_{\mathbb{C}}^{n-k}(\mathcal{M})$ such that for all $x \in \mathcal{M}$ and $\eta \in \Omega_{\mathbb{C}}^{n-k}(\mathcal{M})$

$$(\overline{\omega} \wedge \eta)(x) = \langle \eta(x), (*\omega)(x) \rangle_{\bigwedge_{\mathbb{C}}^{n-k} T_x^* \mathcal{M}} \, \mathrm{dvol}(x)\,.$$

In fact, one can show that

$$d^*|_{\Omega_{\mathbb{C}}^{k}(\mathcal{M})} = (-1)^{nk+1} * d * .$$

That $\mathcal{D}_0$ is essentially selfadjoint follows from results about general symmetric differential operators on manifolds (based on Friedrichs modifiers). In order to verify that $\mathcal{D}$ has compact resolvents, one defines the Sobolev spaces

$$\mathcal{H}_s := \{\omega \in \mathcal{H} \mid (1+\Delta)^{s/2}\omega \in \mathcal{H}\}$$

for $s \geq 0$ and uses the Rellich Lemma to show that $H_1 \hookrightarrow H_0$ is compact, which implies that

$$(1+\Delta)^{-1/2} \colon \mathcal{H} = \mathcal{H}_0 \longrightarrow \mathcal{H}_1 \longrightarrow \mathcal{H}_0 = \mathcal{H}$$

is compact and hence $(\mathcal{D} - \mathrm{i}1)^{-1}$ is compact.

Remark 2.22: If the manifold $\mathcal{M}$ carries more structure (i.e., spin^c-*manifold*), there is another spectral triple $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ associated to $\mathcal{M}$, with $\mathcal{D}$ being the *Dirac operator*. We do not go into details here.

Exercises

Exercise 2.1: (i) Let $x_0 \in \mathbb{R}^n$ be given. For $j = 1, \ldots, n$, we define a linear map $\partial_j|_{x_0} \colon C^\infty_{x_0}(\mathbb{R}^n) \to \mathbb{R}$ by $\partial_j|_{x_0}([f]_{x_0}) := (\partial_j f)(x_0) = \frac{\partial f}{\partial x_j}(x_0)$ for every germ $[f]_{x_0} \in C^\infty_{x_0}(\mathbb{R}^n)$. Prove that $\{\partial_j|_{x_0} \mid j = 1, \ldots, n\}$ forms a basis of the tangent space $T_{x_0}\mathbb{R}^n$.

(ii) Let $\mathcal{M}$ be a n-dimensional smooth manifold with the maximal smooth atlas $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$. Show that for every $i \in I$ and each $x_0 \in U_i$, the linear map

$$\Theta_{i,x_0} \colon \mathbb{R}^n \longrightarrow T_{x_0}\mathcal{M}$$

that is defined by

$$\Theta_{i,x_0}(v)\Big([f]_{x_0}\Big) := \sum_{j=1}^{n} v^j \Big(\partial_j(f \circ \phi_i^{-1})\Big)(\phi_i(x_0))$$

for each $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ and every germ $[f]_{x_0} \in C^\infty_{x_0}(\mathcal{M})$, is an isomorphism of real vector spaces.

Exercise 2.2: (i) Let E be an n-dimensional (real or complex) vector bundle over a Hausdorff topological space X.

Construct an n-dimensional (real or complex) vector bundle E^* over X, such that for each $x \in X$ the fibre E^*_x of E^* is the dual space of the fibre E_x of E, i.e., $E^*_x = \hom(E_x, \mathbb{K})$.

We call E^* the *dual bundle of E*.

(ii) Let E be an n-dimensional smooth (real or complex) vector bundle over a smooth manifold $\mathcal{M}$. Show that the dual bundle E^* of E is also smooth.

Exercise 2.3: Complete the proof of Theorem 2.9 of the lecture by proving the following assertions for a smooth manifold $\mathcal{M}$ of dimension n and an open subset $V \subseteq \mathcal{M}$:

(i) For every $D \in \operatorname{der} C^\infty(V)$, the map $\Psi(D)\colon V \to T\mathcal{M}, x \mapsto (\Psi(D))(x)$ belongs to $\mathfrak{X}(V)$. Recall that $(\Psi(D))(x_0) \in T_{x_0}\mathcal{M}$ for any point $x_0 \in V$ is defined by

$$(\Psi(D))(x_0)\colon C^\infty_{x_0}(\mathcal{M}) \longrightarrow \mathbb{R}, \qquad [f]_{x_0} \longmapsto D|_{x_0}(\rho \cdot f|_V),$$

where $\rho\colon \mathcal{M} \to [0,1]$, for a chosen representative (U, f) of the given germ $[f]_{x_0}$, is a bump function for (U, x_0).

(ii) The induced linear map $\Psi\colon \operatorname{der} C^\infty(V) \to \mathfrak{X}(V), D \mapsto \Psi(D)$ satisfies

$$\Phi \circ \Psi = \mathrm{id}_{\operatorname{der} C^\infty(V)} \qquad \text{and} \qquad \Psi \circ \Phi = \mathrm{id}_{\mathfrak{X}(V)},$$

where $\Phi\colon \mathfrak{X}(V) \to \operatorname{der} C^\infty(V)$ is the linear map defined in Theorem 2.9.

Exercise 2.4: Let $\mathcal{M}$ be an oriented paracompact smooth manifold of dimension n and let g be a Riemannian metric on $\mathcal{M}$. Show that for every $f \in C^\infty_c(\mathcal{M})$ with the property that $\operatorname{supp}(f) \subset U$ for some local chart (U, φ) in the maximal oriented smooth atlas $\mathcal{A}$ of $\mathcal{M}$, the value $\int_{\mathcal{M}} f$ that is assigned to f by formula (2.1) of the lecture, does not depend on the particular choice of (U, φ).

Exercise 2.5: Let $\mathcal{M}$ be an oriented compact smooth manifold of dimension n and let g be a Riemannian metric on $\mathcal{M}$. Prove the identity

$$\langle d\overline{f} \wedge \eta, \omega \rangle_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})} = \langle \eta, df \llcorner \omega \rangle_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})}$$

for all $f \in C^\infty(\mathcal{M}, \mathbb{C})$ and all $\omega, \eta \in \Omega^\bullet_{\mathbb{C}}(\mathcal{M})$.

Chapter 3.

The geodesic distance in noncommutative geometry

Associating a spectral triple such as the Hodge-de Rham triple to a compact oriented smooth Riemannian manifolds follows the philosophy of "spectral geometry", where such classical geometric objects are studied by spectral properties of canonically associated differential operators. At the same time, this allows to carry classical concepts over to a noncommutative setting.

In this chapter, we discuss the geodesic distance within the framework of spectral triples.

Definition 3.1 (Geodesic distance): Let $(\mathcal{M}, g)$ be a *Riemannian manifold*, i.e, $\mathcal{M}$ is a paracompact smooth manifold with Riemannian metric g, and let $x_0, x_1 \in \mathcal{M}$. Then

(i) We denote by $\Gamma(x_0, x_1)$ the set of all smooth paths $\gamma\colon [0,1] \to \mathcal{M}$ satisfying $\gamma(0) = x_0$, $\gamma(1) = x_1$. Note that $\mathcal{M}$ is *connected* if and only if $\Gamma(x_0, x_1)$ is non-empty for every choice of points $x_0, x_1 \in \mathcal{M}$.

(ii) If $\gamma \in \Gamma(x_0, x_1)$ is given, we define $\gamma'(t) \in T_{\gamma(t)}\mathcal{M}$ for each $t \in [0,1]$ by $\gamma'(t)([f]_{\gamma(t)}) := (f \circ \gamma)'(t)$ for all $[f]_{\gamma(t)} \in C^\infty_{\gamma(t)}(\mathcal{M})$.[1] The *length* $L(\gamma)$ of γ is then defined as

$$L(\gamma) := \int_0^1 g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}\, dt$$

(iii) Suppose that $\mathcal{M}$ is connected. The *geodesic distance* $d_g(x_0, x_1)$ between x_0 and x_1 is defined as

$$d_g(x_0, x_1) := \inf\{L(\gamma) \mid \gamma \in \Gamma(x_0, x_1)\}.$$

[1]It happend in the same way in Remark 2.3.

Remark 3.2: (i) On a connected Riemannian manifold $(\mathcal{M}, g)$, the geodesic distance induces a metric

$$d_g\colon \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty), \qquad (x_0, x_1) \longmapsto d_g(x_0, x_1)$$

called the *Riemannian distance function.* Note that if $x_0 \neq x_1$, the geodesic distance $d_g(x_0, x_1)$ is a positive number. Indeed, for $\gamma \in \Gamma(x_0, x_1)$ and a local chart (U, φ) with $x_0 \in U$ and $x_1 \notin U$, we have for all $t \in [0, T]$ the equality

$$g_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \langle G(\gamma(t))v'(t), v'(t)\rangle$$

where $G = (g_{k,l})_{1\leq k,l\leq n}\colon \varphi(U) \longrightarrow M_n(\mathbb{R})$ defined for all $x \in U$ by

$$g_{k,l}(\varphi(x)) := g_x\Big((d\varphi)(x)^{-1}(\partial_k|_{\varphi(x)}), (d\varphi)(x)^{-1}(\partial_l|_{\varphi(x)})\Big)$$

(see Theorem 2.17) and the smooth map $v\colon [0,T] \to \mathbb{R}^n, t \mapsto \varphi(\gamma(t))$, where $T \in [0, 1]$ is chosen such that $\gamma([0, T]) \subset U$.

Take $r > 0$ such that $\mathrm{cl}(B(\varphi(x_0), r)) \subset \varphi(U)$ and $V := \varphi^{-1}(B(\varphi(x_0), r))$ which is an open subset of U. We find $\delta \in (0, 1]$ such that for all $y \in B(\varphi(x_0), r)$ and $\xi \in \mathbb{R}^n$ it holds

$$\delta\|\xi\| \leq \langle G(y)\xi, \xi\rangle^{1/2} \leq \delta^{-1}\|\xi\|.$$

Thus

$$\begin{aligned} L(\gamma) &\geq \int_0^{T'} g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}\, dt \\ &= \int_0^{T'} \langle G(\gamma(t))v'(t), v'(t)\rangle^{1/2}\, dt \\ &\geq \delta \int_0^{T'} \|v'(t)\|\, dt \geq \delta \left\| \int_0^{T'} v'(t)\, dt \right\| = \delta\|v(T') - \varphi(x_0)\| \end{aligned}$$

for every $T' \in (0, T]$ with $\gamma([0, T']) \subset V$. By enlarging T and taking the limit in T', we infer that $L(\gamma) \geq \delta r > 0$ and thus we have $d_g(x_0, x_1) \geq \delta r > 0$.

(ii) The topology on $\mathcal{M}$ induced by the metric d_g agrees with the given topology on $\mathcal{M}$. This can be shown by arguments similar to (i). In fact, one shows that for each $x_0 \in \mathcal{M}$ and a local chart (U, φ) with $x_0 \in U$, an open neighbourhood $V \subseteq U$ of x_0 and $\delta \in (0, 1]$, it exists $r > 0$ such that for all $x \in V$ it holds

$$\delta|\varphi(x) - \varphi(x_0)| \leq d_g(x, x_0) \leq \delta^{-1}|\varphi(x) - \varphi(x_0)|,$$

and $d_g(x, x_0) \geq \delta r$ for all $x \in \mathcal{M} - V$.

(iii) A connected paracompact smooth manifold is *second countable* (i.e., admits a countable base). Thus, it follows from the Urysohn metrisation theorem that the topology on $\mathcal{M}$ must be metrisable; this is in keeping with (ii).

(iv) Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two connected Riemannian manifolds. Then every isometry $\varphi\colon (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$, i.e., a map $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ that satisfies for all $x_0, x_1 \in \mathcal{M}_1$ that

$$d_{g_2}(\varphi(x_0), \varphi(x_1)) = d_{g_1}(x_0, x_1)$$

is necessarily smooth and satisfies $\varphi^* g_2 = g_1$ — this is the result of the Myers-Steenrod theorem from 1939.

Note that if $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a smooth immersion between (paracompact) smooth manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ and if g is a Riemannian metric on $\mathcal{M}_2$, then $\varphi^* g$ is the Riemannian metric on $\mathcal{M}_1$ given by

$$(\varphi^* g)_x(\alpha, \beta) := g_{\varphi(x)}\Big((d\varphi)(x)(\alpha), (d\varphi)(x)(\beta)\Big)$$

for all $x \in \mathcal{M}_1$ and $\alpha, \beta \in T_x\mathcal{M}_1$, where $(d\varphi)(x)\colon T_x\mathcal{M}_1 \to T_x\mathcal{M}_2$ is defined as in (Theorem 2.17); φ is called an immersion, if $(d\varphi)(x)$ is injective for each $x \in \mathcal{M}$.

Our goal is to "dualise" the definition of the geodesic distance such that it fits into the framework of spectral triples.

Theorem 3.3 (Musical isomorphisms): *Let $(\mathcal{M}, g)$ be a Riemannian manifold. Then g can be seen as a positive definite pairing on smooth vector fields, i.e., a map*

$$g\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$$

which is $C^\infty(\mathcal{M})$-bilinear and satisfies $g(X, X) \geq 0$ for all $X \in \mathfrak{X}(\mathcal{M})$ and $g(X, X)(x) = 0$ at $x \in \mathcal{M}$ if and only if $X(x) = 0$.

This induces an isomorphism (in fact, a $C^\infty(\mathcal{M})$-bimodule map)

$$\flat\colon \mathfrak{X}(\mathcal{M}) \longrightarrow \Omega^1(\mathcal{M}), \qquad X \longmapsto X^\flat := g(X, \cdot).$$

Its inverse $\sharp\colon \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, $\omega \mapsto \omega^\sharp$ is determined by $\omega(X) = g(\omega^\sharp, X)$ for all $X \in \mathfrak{X}(\mathcal{M})$. The inner product on $\Omega^1(\mathcal{M})$ defined in (Remark 2.18) (iii) satisfies

$$\langle \omega, \eta \rangle_{\Omega^1(\mathcal{M})} = \int_{\mathcal{M}} g(\omega^\sharp, \eta^\sharp)$$

for all ω, $\eta \in \Omega^1(\mathcal{M}$, since $\langle \omega(x), \eta(x) \rangle_{T_x^*(\mathcal{M})} = g_x(\omega^\sharp(x), \eta^\sharp(x))$ for each $x \in \mathcal{M}$.

Definition 3.4: Let $(\mathcal{M}, g)$ be a Riemannian manifold. The *gradient* $\operatorname{grad} f$ of a function $f \in C^\infty(\mathcal{M})$ is the vector field

$$\operatorname{grad} f := (d^0 f)^\sharp \in \mathfrak{X}(\mathcal{M})$$

with $d^0 f \in \Omega^1(\mathcal{M})$ as defined in (Definition 2.14) (iii). We thus have that for each $x \in \mathcal{M}$ and $\delta \in T_x\mathcal{M}$

$$g_x(\operatorname{grad}_x f, \delta) = (d^0 f)(x)(\delta). \tag{3.1}$$

Definition 3.5: Let $(\mathcal{M}, g)$ be a compact Riemannian manifold. On $\mathfrak{X}(\mathcal{M})$, we define a norm $\|\cdot\|_\infty$ by

$$\|X\|_\infty := \max_{x \in \mathcal{M}} g_x(X(x), X(x)) \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}).$$

Theorem 3.6: *Let $(\mathcal{M}, g)$ be a compact and connected Riemannian manifold. Then, for all x_0, $x_1 \in \mathcal{M}$, we have that*

$$d_g(x_0, x_1) = \sup\{|f(x_1) - f(x_0) \mid f \in C^\infty(\mathcal{M}) : \|\operatorname{grad} f\|_\infty \leq 1\}.$$

Proof: Take any $\gamma \in \Gamma(x_0, x_1)$. Then, for every $f \in C^\infty(\mathcal{M})$, it holds

$$f(x_1) - f(x_0) = f(\gamma(1)) - f(\gamma(0)) = \int_0^1 (f \circ \gamma)'(t)\, dt$$

and for each $t \in [0, 1]$, we have

$$(f \circ \gamma)'(t) = (d^0 f)(\gamma(t))(\gamma'(t)) = g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \gamma'(t)),$$

so that the Cauchy-Schwarz inequality yields

$$\begin{aligned} |(f \circ \gamma)'(t)| &\leq g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \operatorname{grad}_{\gamma(t)} f)^{1/2} g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2} \\ &\leq \|\operatorname{grad} f\|_\infty g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}. \end{aligned}$$

In summary, we get $|f(x_1) - f(x_0)| \leq \|\operatorname{grad} f\|_\infty L(\gamma)$. We infer from the latter that

$$\sup\{|f(x_1) - f(x_0) \mid f \in C^\infty(\mathcal{M}) : \|\operatorname{grad} f\|_\infty \leq 1\} \leq d_g(x_0, x_1).$$

In order to prove "$\geq$", we consider the function $f_0 \colon \mathcal{M} \to \mathbb{R}$, $x \mapsto d(x_0, x)$. While f_0 is not (necessarily) smooth, the triangle inequality for d_g impiles that f_0 is at least Lipschitz continuous with Lipschitz constant 1. For every $\varepsilon > 0$, we find $h_\varepsilon \in C^\infty(\mathcal{M})$ such that $\|f_0 - h_\varepsilon\|_\infty < \varepsilon$ and $\|\operatorname{grad} h_\varepsilon\|_\infty \leq 1+\varepsilon$; put $f_\varepsilon := \frac{1}{1+\varepsilon} h_\varepsilon \in C^\infty(\mathcal{M})$. Then $\|\operatorname{grad} f_\varepsilon\|_\infty \leq 1$ and it holds

$$\begin{aligned}
|f_\varepsilon(x_1) - f_\varepsilon(x_0)| &= \frac{1}{1+\varepsilon}|h_\varepsilon(x_1) - h_\varepsilon(x_0)| \\
&= \frac{1}{1+\varepsilon}|f_0(x_1) - (f_0(x_1) - h_\varepsilon(x_1)) + (h_\varepsilon(x_0) - f_0(x_0))| \\
&\geq \frac{1}{1+\varepsilon}\Big(|f_0(x_1)| - (|f_0(x_1) - h_\varepsilon(x_1)| + |f_0(x_0) - h_\varepsilon(x_0)|\Big) \\
&\geq \frac{1}{1+\varepsilon} d_g(x_0, x_1) - \frac{2\varepsilon}{1+\varepsilon},
\end{aligned}$$

which shows that for every $\varepsilon > 0$ it holds

$$\sup\{|f(x_1) - f(x_0)| \mid f \in C^\infty(\mathcal{M}) : \|\operatorname{grad} f\|_\infty \leq 1\} \geq \frac{d_g(x_0, x_1)}{1+\varepsilon} - \frac{2\varepsilon}{1+\varepsilon}.$$

We conclude by taking the limit $\varepsilon \downarrow 0$. $\square$

The preceeding motivates the following definition:

Definition 3.7: Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. We define by

$$A := \operatorname{cl}_{\|\cdot\|_\infty}(\pi(\mathcal{A})) \subseteq B(\mathcal{H})$$

a C^*-algebra and denote by $S(A)$ the state space of A. We define for $\varphi, \psi \in S(A)$ by

$$d_\mathcal{D}(\varphi, \psi) := \sup\{|\psi(\pi(a)) - \varphi(\pi(a))| \mid a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\| \leq 1\} \in [0, \infty]$$

the *spetral distance* between φ and ψ.

In view of Remark 3.2 (iv), the following theorem says that the Hodge-de Rham triple remembers the metric.

Theorem 3.8: *Let $(\mathcal{M}, g)$ be a compact oriented Riemannian manifold and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the Hodge-de Rham triple for $(\mathcal{M}, g)$. Then the faithful representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ extends to a faithful *-representation $\hat{\pi}\colon C(\mathcal{M}, \mathbb{C}) \to B(\mathcal{H})$ which induces an isometric *-isomorphism*

$$\hat{\pi}\colon C(\mathcal{M}, \mathbb{C}) \xrightarrow{\cong} A := \operatorname{cl}_{\|\cdot\|}(\mathcal{A}) \subseteq B(\mathcal{H}).$$

Define $\delta_x \in S(A)$ for $x \in \mathcal{M}$ by $\delta_x(\hat{\pi}(f)) := f(x)$ for $f \in C(\mathcal{M}, \mathbb{C})$. If $\mathcal{M}$ is commutative, then $d_g(x_0, x_1) = d_\mathcal{D}(\delta_{x_0}, \delta_{x_1})$ for all $x_0, x_1 \in \mathcal{M}$.

Proof: (1) By definition of $\mathcal{H}$, it is easily seen that each $f \in C(\mathcal{M}, \mathbb{C})$ defines an operator $\widehat{\pi}(f) \in B(\mathcal{H})$ by $\widehat{\pi}(f)\omega := f\omega$ for all $\omega \in \mathcal{H}$ with $\|\widehat{\pi}(f)\| \leq \|f\|_\infty$; in fact, we have that $\|\widehat{\pi}(f)\| = \|f\|_\infty$, since $\widehat{\pi}(f)$ restricts to the ordinary multiplication operator on $L^2(\mathcal{M}, g) := \mathrm{cl}(\Omega^0_{\mathbb{C}}(\mathcal{M}))$, for which we know $\|\widehat{\pi}(f)|_{L^2(\mathcal{M},g)}\| = \|f\|_\infty$. Thus, $\widehat{\pi}\colon C(\mathcal{M}, \mathbb{C}) \to B(\mathcal{H})$, $f \mapsto \widehat{\pi}(f)$ is a faithful (isometric) *-representation, which extends $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and induces an isometric *-isomorphism $C(\mathcal{M}, \mathbb{C}) \cong A$.

(2) By equation (Eq. 2.2) in (Remark 2.20), we have for $f = \overline{f} \in \mathcal{A}$, and forms $\omega \in \Omega^\bullet_{\mathbb{C}}(\mathcal{M})$, and each point $x \in \mathcal{M}$ that

$$([\mathcal{D}, \pi(f)]\omega)(x) = (df)(x) \bullet \omega(x) =: c((df)(x))\omega(x),$$

where, for each real $v \in T^*_x\mathcal{M}_{\mathbb{C}}$, the operator

$$c(v)\colon \bigwedge\nolimits^{\bullet}_{\mathbb{C}} T^*_x\mathcal{M}_{\mathbb{C}} \longrightarrow \bigwedge\nolimits^{\bullet}_{\mathbb{C}} T^*_x\mathcal{M}_{\mathbb{C}}, \qquad \omega \longmapsto v \cdot \omega$$

is an isometry (refer to (Exercise 3B 1(ii))). Hence

$$\|([\mathcal{D}, \pi(f)]\omega)(x)\|_{\bigwedge^\bullet_{\mathbb{C}} T^*_x\mathcal{M}_{\mathbb{C}}} \leq \|(df)(x)\|_{T^*_x\mathcal{M}_{\mathbb{C}}} \cdot \|\omega(x)\|_{\bigwedge^\bullet_{\mathbb{C}} T^*_x\mathcal{M}_{\mathbb{C}}}.$$

We conclude that

$$\|[\mathcal{D}, \pi(f)]\omega\|_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})} \leq \left(\max_{x\in\mathcal{M}} \|(df)(x)\|_{T^*_x\mathcal{M}_{\mathbb{C}}}\right) \cdot \|\omega\|_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})}$$

and so, by (Definition 3.4) and (Definition 3.5),

$$\|[\mathcal{D}, \pi(f)]\| \leq \max_{x\in\mathcal{M}} \|(df)(x)\|_{T^*_x\mathcal{M}_{\mathbb{C}}} = \|\mathrm{grad}\, f\|_\infty.$$

Optimising ω, we get that $\|[\mathcal{D}, \pi(f)]\| = \|\mathrm{grad}\, f\|_\infty$ and thus, by (Theorem 3.6) and (Exercise 4AB - 1(ii))

$$\begin{aligned} d_g(x_0, x_1) &= \sup\{|f(x_1) - f(x_0)| \mid f \in C^\infty(\mathcal{M}) : \|\mathrm{grad}\, f\|_\infty \leq 1\} \\ &= \sup\{|\delta_{x_1}(f) - \delta_{x_0}(f)| \mid f = \overline{f} \in \mathcal{A} : \|[D, \pi(f)]\| \leq 1\} = d_{\mathcal{D}}(\delta_{x_0}, \delta_{x_1}) \end{aligned}$$

which concludes the proof. $\square$

Exercises

Exercise 3.1: Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with the faithful *-representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$. Consider the state space $S(A)$ for the associated C^*-algebra $A := \mathrm{cl}_{\|\cdot\|}(\pi(\mathcal{A})) \subseteq B(\mathcal{H})$. Prove the following assertions:

(i) If the image of the set $\{a \in \mathcal{A} \mid \|[\mathcal{D}, \pi(a)]\| \leq 1\}$ under the canonical projection in the quotient Banach space $A/\mathbb{C}1$ is a norm bounded set, then the spectral distance satisfies $d_{\mathcal{D}}(\phi, \psi) < \infty$ for all $\phi, \psi \in S(A)$ and induces a metric $d_{\mathcal{D}}\colon S(A) \times S(A) \to [0, \infty)$.

(ii) For all $\phi, \psi \in S(A)$, we have that

$$d_{\mathcal{D}}(\phi, \psi) = \sup\{|\psi(\pi(a)) - \phi(\pi(a))| \mid a = a^* \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\| \leq 1\}.$$

Hint: To prove "$\leq$", establish first that the set $\{a \in \mathcal{A} \mid \|[\mathcal{D}, \pi(a)]\| \leq 1\}$ is closed under the following maps: $a \mapsto \zeta a$ for each $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $a \mapsto a^*$, $a \mapsto \mathrm{Re}(a) = \frac{1}{2}(a + a^*)$, and $a \mapsto \mathrm{Im}(a) = \frac{1}{2\mathrm{i}}(a - a^*)$.

Exercise 3.2: Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples with the faithful *-representations $\pi_1 : \mathcal{A}_1 \to B(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A}_2 \to B(\mathcal{H}_2)$, respectively.

We call these two spectral triples *equivalent*, if there exists a *-isomorphism $\Phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ and a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $a \in \mathcal{A}_1$ it holds

$$U\pi_1(a)U^* = \pi_2(\Phi(a))$$

and $U\mathcal{D}_1U^* = \mathcal{D}_2$.

Show that in this case $\mathrm{ad}_U\colon B(\mathcal{H}_1) \to B(\mathcal{H}_2), x \mapsto UxU^*$ is an isometry which satisfies $\mathrm{ad}_U(A_1) = A_2$, where A_1 and A_2 are the C^*-algebras associated to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and prove that $\mathrm{ad}_U^*\colon S(A_2) \to S(A_1), \phi \mapsto \phi \circ \mathrm{ad}_U$ defines an isometry for the spectral distances, i.e.,

$$d_{\mathcal{D}_1}(\mathrm{ad}_U^* \phi, \mathrm{ad}_U^* \psi) = d_{\mathcal{D}_2}(\phi, \psi) \qquad \text{for all } \phi, \psi \in S(A_2).$$

Exercise 3.3: Consider the complex unital *-algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ with entry-wise operations. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional complex Hilbert spaces and put $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Define the *-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$ for all $a = (a_1, a_2) \in \mathcal{A}$ by

$$\pi(a) := \begin{pmatrix} a_1 \,\mathrm{id}_{\mathcal{H}_1} & 0 \\ 0 & a_2 \,\mathrm{id}_{\mathcal{H}_2} \end{pmatrix}.$$

Further, take any linear operator $M\colon \mathcal{H}_1 \to \mathcal{H}_2$ and consider the operator

$$\mathcal{D} := \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

(i) Verify that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple. Compute for all $a = (a_1, a_2) \in \mathcal{A}$ the commutator $[\mathcal{D}, \pi(a)]$ and show that its norm is given by

$$\|[\mathcal{D}, \pi(a)]\| = |a_2 - a_1|\|M\|.$$

(ii) Consider the states $\delta_1, \delta_2 \colon \mathcal{A} \to \mathbb{C}$ that are respectively given by $\delta_1(a) = a_1$ and $\delta_2(a) = a_2$ for each $a = (a_1, a_2) \in \mathcal{A}$. Compute the spectral distance $d_{\mathcal{D}}(\delta_1, \delta_2)$.

(iii) Show that the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *even*, i.e., there is a selfadjoint operator $\Gamma \in B(\mathcal{H})$ with the properties that $\Gamma^2 = \mathrm{id}_{\mathcal{H}}$, $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$, and $\pi(a)\Gamma = \Gamma\pi(a)$ for all $a \in \mathcal{A}$. We call Γ a *grading* on $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Chapter 4.

The Riemann-Lebesgue measure in noncommutative geometry

In (Chapter 3), we have seen that the geodesic distance on a connected, compact and oriented Riemannian manifold can be recovered from its associated Hodge-de Rham triple via Connes spectral distance.

In this chapter, we will discuss the noncommutative integral, by which integration of (smooth) functions with respect to the *Riemann-Lebesgue measure* on Riemannian manifolds as introduced in (Theorem 2.17) is generalised to the framework of spectral triples.

Like in quantum mechanics, the underlying idea is that operators on a separable complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = \infty$ take over the role of complex variables, while selfadjoint operators on $\mathcal{H}$ correspond to real variables.

Remark 4.1: Recall (Theorem 9.8 from the Functional Analysis I lecture notes) that $T \in B(\mathcal{H})$ is compact if and only if T can be approximated in operator norm on $B(\mathcal{H})$ by finite rank operators; equivalently,

$$\forall\, \varepsilon > 0\, \exists\, V \subseteq \mathcal{H} \text{ subspace}, \dim V < \infty : \|T|_{V^\perp}\| < \varepsilon,$$

where $T|_{V^\perp} \colon V^\perp \to \mathcal{H}$ is the restriction of T to $V^\perp$ and $\|\cdot\|$ is the norm on $B(V^\perp, \mathcal{H})$.

For a compact operator $T \in B(\mathcal{H})$, we call the non-zero eigenvalues $(\mu_n(T))_{n\geq 0}$ of $|T| := (T^*T)^{1/2}$, arragned in decreasing order and repeated according to multiplicity, the *characteristic values of* T. Note that $\mu_n(T)$ converges to 0 as $n \to \infty$. We have that for all $n \in \mathbb{N}_0$

$$\begin{aligned} \mu_n(T) &= \inf\{\|T - S\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran} S \leq n\} \\ &= \inf\{\|T|_{V^\perp}\| \mid V \subseteq \mathcal{H} \text{ subspace}, \dim V = n\} \end{aligned} \tag{4.1}$$

and in particular $\mu_0(T) = \|T\|$.

In view of Remark 4.1, compact operators are considered as "infinitesimals" in our "quantised calculus"; their "size" is measured by the rate of decay of their sequence of characteristic values.

Definition 4.2: Let $T \in K(\mathcal{H})$ and $\alpha > 0$ be given. We say that T is an *infinitesimal of order* α, if $\mu_n(T) = O(n^{-\alpha})$ as $n \to \infty$, i.e., if there is a constant $C < \infty$ such that for all $n \in \mathbb{N}$ it holds $\mu_n(T) \leq Cn^{-\alpha}$. For $\alpha > 0$, we denote by $I_\alpha(\mathcal{H})$ the set of all $T \in K(\mathcal{H})$ which are infinitesimals of order α.

Remark 4.3: Recall (Theorem 9.5 from the Functional Analysis I lecture notes) that $K(\mathcal{H})$ is a norm-closed two-sided ideal in $B(\mathcal{H})$. It follows from Eq. (4.1) that for all $T \in K(\mathcal{H})$ and $S \in B(\mathcal{H})$

$$\mu_n(TS) \leq \|S\|\mu_n(T) \quad \text{and} \quad \mu_n(ST) \leq \|S\|\mu_n(T)$$

and that for T_1, $T_2 \in K(\mathcal{H})$

$$\mu_{n+m}(T_1 + T_2) \leq \mu_n(T_1) + \mu_m(T_2);$$

thus each $I_\alpha(\mathcal{H})$ forms a (non-closed) two-sided ideal in $B(\mathcal{H})$. Furthermore, we have for T_1, $T_2 \in K(\mathcal{H})$ that

$$\mu_{n+m}(T_1T_2) \leq \mu_n(T_1)\mu_m(T_2)$$

which implies the following rule for infinitesimals: If T_1 is of order α_1 and T_2 is of order α_2, then T_1T_2 is of order $\alpha_1 + \alpha_2$.

We want to find an "integral" that is defined on $I_1(\mathcal{H})$ and neglects all infinitesimals of order $\alpha > 1$.

Remark 4.4: An operator $T \in B(\mathcal{H})$ is said to be in *trace class*, if $\sum_{k=0}^\infty \langle |T|\xi_k, \xi_k\rangle$ is finite for some (and in turn for each) orthonormal basis $(\xi_k)_{k\in\mathbb{N}_0}$ of $\mathcal{H}$. In this case, the sum $\sum_{k=0}^\infty \langle T\xi_k, \xi_k\rangle$ is absolutely convergent and its value $\mathrm{Tr}(T)$ is indepentend of the choice of the orthonormal basis $(\xi_k)_{k\in\mathbb{N}_0}$ of $\mathcal{H}$; we call $\mathrm{Tr}(T)$ the *trace* of T.

The set $\mathscr{L}^1(\mathcal{H})$ of all trace class operators on $\mathcal{H}$ forms a (non-closed) two-sided ideal in $\mathcal{B}(H)$. However, $\mathscr{L}^1(\mathcal{H})$ is a Banach space with respect to the norm $\|T\|_1 := \mathrm{Tr}(|T|)$. Note that $\mathscr{L}^1(\mathcal{H}) \subseteq K(\mathcal{H})$. If $T \in \mathscr{L}^1(\mathcal{H})$ is positive, then

$$\mathrm{Tr}(T) = \sum_{n=0}^\infty \mu_n(T). \tag{4.2}$$

Thus, $\mathrm{Tr}\colon \mathscr{L}^1(\mathcal{H}) \to \mathbb{C}$ is not appropriate for our purpose, because:

- $I_1(\mathcal{H})$ is not a subset of $\mathscr{L}^{(1,\infty)}(\mathcal{H})$, so that Tr is not defined on all infinitesimals of order 1.
- Tr does not vanish even on finite rank operators, but those belong to $I_\alpha(\mathcal{H})$ for every $\alpha > 1$.

Definition 4.5: For $T \in K(\mathcal{H})$, we define for each $N \in \mathbb{N}$

$$\sigma_N(T) := \sum_{n=0}^{N-1} \mu_n(T) \quad \text{and} \quad \gamma_N(T) := \frac{1}{\log N}\sigma_N(T).$$

Remark 4.6: Note that $\sigma_N(T)$ is a partial sum in Eq. (4.2). For $T \in I_1(\mathcal{H})$, we find constants $C, C' < \infty$ such that for all $N \in \mathbb{N}$

$$\sigma_N(T) = \|T\| + \sum_{n=1}^{N-1} \frac{C}{n} \leq C' \log(N).$$

Consequently, $(\gamma_N(T))_{N\in\mathbb{N}}$ is a bounded sequence. Thus $I_1(\mathcal{H}) \subseteq \mathscr{L}^{(1,\infty)}(\mathcal{H})$, where

$$\mathscr{L}^{(1,\infty)} := \left\{ T \in K(\mathcal{H}) \;\middle|\; \|T\|_{(1,\infty)} := \sup_{N\in\mathbb{N}} \gamma_N(T) < \infty \right\}$$

is the *Dixmier ideal.* Note that $\mathscr{L}^1(\mathcal{H}) \subseteq \mathscr{L}^{(1,\infty)}(\mathcal{H})$.

Proposition 4.7: *Consider operators T_1, $T_2 \in K(\mathcal{H})$. For all $N \in \mathbb{N}$, we have that*

$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2)$$

and if T_1, T_2 are positive, we have in addition that

$$\sigma_{2N}(T_1 + T_2) \geq \sigma_N(T_1) + \sigma_N(T_2).$$

It follows that for any positive T_1, $T_2 \in K(\mathcal{H})$

$$\gamma_N(T_1 + T_2) \leq \gamma_N(T_1) + \gamma_N(T_2) \leq \gamma_{2N}(T_1 + T_2)\left(1 + \frac{\log 2}{\log N}\right).$$

Proof: Exercise! □

Theorem 4.8 (Dixmier traces): *Let $\omega\colon \ell^\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ be a linear map such that*

(i) $\omega((\alpha_N)_{N\in\mathbb{N}}) \geq 0$ *if* $\alpha_N \geq 0$ *for all* $N \in \mathbb{N}$,

(ii) $\omega((\alpha_N)_{N\in\mathbb{N}}) = \lim_{N\to\infty} \alpha_N$, *if* $(\alpha_N)_{N\in\mathbb{N}}$ *is convergent,*

(iii) $\omega((\alpha_{2N})_{N\in\mathbb{N}}) = \omega((\alpha_N)_{N\in\mathbb{N}})$ *for each* $(\alpha_N)_{N\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{R})$.[1]

Then, $\mathrm{Tr}_\omega\colon \mathscr{L}^{(1,\infty)}(\mathcal{H}_+ \to [0,\infty)$ *for all* $\lambda \geq 0$, T, T_1, $T_2 \in \mathscr{L}_+^{(1,\infty)}$ *satisfies the conditions*

$$\mathrm{Tr}_\omega(T_1+T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2), \qquad \mathrm{Tr}_\omega(\lambda T) = \lambda\,\mathrm{Tr}_\omega(T)$$

and extends uniquely to a positive linear map $\mathrm{Tr}_\omega\colon \mathscr{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C}$, *which for all* $S \in B(\mathcal{H})$, $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ *and* $U \in \mathscr{L}^1(\mathcal{H}) \subseteq \mathscr{L}^{(1,\infty)}(\mathcal{H})$ *satisfies*

$$\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS) \tag{4.3}$$

and

$$\mathrm{Tr}_\omega(U) = 0. \tag{4.4}$$

Note that $\bigcup_{\alpha>1} I_\alpha(\mathcal{H}) \subseteq \mathscr{L}^1(\mathcal{H})$. *We call* Tr_ω *a* Dixmier trace*.*

Proof: For every $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$, $(\gamma_N(T))_{N\in\mathbb{N}}$ is bounded (see Remark 4.6); hence, $\mathrm{Tr}_\omega(T)$ is well-defined and clearly $\mathrm{Tr}_\omega(T) \geq 0$ by (i). For T_1, $T_2 \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$, it follows from Proposition 4.7 and property (i) of ω that

$$\begin{aligned}\mathrm{Tr}_\omega(T_1+T_2) &\leq \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)\\ &\leq \omega\Big((\gamma_{2N}(T_1+T_2)_{N\in\mathbb{N}}\Big) + \omega\left(\left(\frac{\log(2)}{\log(N)}\gamma_{2N}(T_1+T_2)\right)_{N\in\mathbb{N}}\right).\end{aligned}$$

By property (ii) of ω, we get that

$$\omega\left(\left(\frac{\log(2)}{\log(N)}\gamma_{2N}(T_1+T_2)\right)_{N\in\mathbb{N}}\right) = \lim_{N\to\infty}\frac{\log(2)}{\log(N)}\gamma_{2N}(T_1+T_2) = 0,$$

since $\gamma_{2N}(T_1+T_2)$ is bounded and property (iii) yields that

$$\omega\Big((\gamma_{2N}(T_1+T_2)_{N\in\mathbb{N}}\Big) = \omega\omega\Big((\gamma_N(T_1+T_2)_{N\in\mathbb{N}}\Big) = \mathrm{Tr}_\omega(T_1+T_2).$$

In summary, we get that $\mathrm{Tr}_\omega(T_1+T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$. That for all $\lambda \geq 0$ and $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$ it holds $\mathrm{Tr}_\omega(\lambda T) = \lambda\,\mathrm{Tr}_\omega(T)$ is clear since ω is linear and $\gamma_N(\lambda T) = \lambda\gamma_N(T)$ for each $N \in \mathbb{N}$.

The extension of Tr_ω to $\mathscr{L}^{(1,\infty)}(\mathcal{H})$ uses that

[1] This condition is called the *scale invariance*.

• each $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ can be written uniquely as $T = \Re(T) + \mathrm{i}\Im(T)$ with selfadjoint $\Re(T)$, $\Im(T) \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$;

• each $T = T^* \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ can be written as $T = P_+|T|P_+ - P_-|T|P_-$ for projections P_+, $P_- \in \mathrm{vN}(T)$ satisfying $P_+ - P_- = 1$ (using the measurable functional calculus $\tilde{\Phi}\colon B_b(\mathrm{Sp}(T)) \to \mathrm{vN}(T)$ from Remark 6.12 (iv), Functional Analysis II, with $P_+ := \tilde{\Phi}(\chi_{[0,\infty)})$ and $P_- := \tilde{\Phi}(\chi_{(-\infty,0)})$, where $P_+|T|P_+$, $P_-|T|P_- \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$ due to Remark 4.3 since $|T| \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$.

Let $U \in B(\mathcal{H})$ be unitary. To prove Eq. (4.3), we note first that Eq. (4.1) implies for $n \in \mathbb{N}_0$ and for all $T \in K(\mathcal{H})$ that $\mu_n(UTU^*) = \mu_n(T)$, which yields by definition for $N \in \mathbb{N}$ that for all $T \in K(\mathcal{H})$ it holds $\gamma_N(UTU^*) = \gamma_N(T)$ and hence it holds for all $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ that $\mathrm{Tr}_\omega(UTU^*) = \mathrm{Tr}_\omega(T)$. Because $\mathscr{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal, this is equivalent to the statement that for all $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ it holds $\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU)$.

Since every $S \in B(\mathcal{H})$ is a linear combination of (in fact four) unitaries (see the proof of Lemma 6.13 in the Functional Analysis II lecture notes), the latter yields Eq. (4.3).

To verify Eq. (4.4), we take $T \in \mathscr{L}^1(\mathcal{H})$ and without loss of generality, we may assume that $T \geq 0$. Note that $(\sigma_N(T))_{n\in\mathbb{N}}$ is bounded by $\|T\|_1$ due to Eq. (4.2), thus $\gamma_N(T) \to 0$ as $N \to \infty$, so that by property (iii) we have

$$\mathrm{Tr}_\omega(T) = \omega\Big((\gamma_N(T))_{N\in\mathbb{N}}\Big) = \lim_{N\to\infty} \gamma_N(T) = 0.$$

Note that $I_\alpha(\mathcal{H}) \subseteq \mathscr{L}^1(\mathcal{H})$ for all $\alpha > 1$, since for each $T \in I_\alpha(\mathcal{H})$, we find $C < \infty$ such that

$$\|T|_1 = \sum_{n=0}^{\infty} \mu_n(T) \leq \|T\| + C \sum_{n=1}^{\infty} n^{-\alpha} < \infty.$$

□

Definition 4.9: Let $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$. We say that T is *measurable* of the value of $\mathrm{Tr}_\omega(T)$ is independent of ω; we denote this common value by $⨍\, T$ and call it the *noncommutative integral of* T. Moreover, we put

$$\mathcal{M}(\mathcal{H}) := \{T \in \mathscr{L}^{(1,\infty)}(\mathcal{H}) \mid T \text{ measurable}\}.$$

Remark 4.10: (i) The existence of (in fact infinitely many) linear maps $\omega\colon \ell^\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ satisfying the conditions (i), (ii) and (iii) in Theorem 4.8 was proved by Dixmier in 1966: With the construction of Tr_ω, he proved the existence of *singular traces on* $B(\mathcal{H})$ (i.e., traces that vanish on $\mathscr{L}^1(\mathcal{H})$) and settled to the negative the question of the uniqueness of the trace on $B(\mathcal{H})$.

(ii) An alternative approach was developed by Connes. It relies on the (piecewise linear) interpolation of $(\sigma_N(T))_{N\in\mathbb{N}}$ given by

$$\sigma_\lambda(T) := \inf\{\|R\|_1 + \lambda\|S\| \mid R \in \mathscr{L}^1(\mathcal{H}), S \in K(\mathcal{H}) : T = R + S\}$$

for each $\lambda > 0$. For $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ and any $a > e$,

$$\gamma\colon [a, \infty) \longrightarrow \mathbb{R}, \qquad \lambda \longmapsto \frac{\sigma_\lambda(T)}{\log(\lambda)}$$

is a continuous and bounded function, its *Cesàro mean* with respect to the Haar measure $\frac{du}{u}$ on the multiplicative group $(0, \infty)$ is for each $\lambda \in [0, \infty)$ given by

$$\tau_\lambda(T) := \frac{1}{\log(\lambda)} \int_a^\lambda \gamma(u)\, \frac{du}{u}$$

and defines a function $\lambda \mapsto \tau_\lambda(T)$ in $C_b([a, \infty))$. For $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})_+$, let $\dot{\tau}(T)$ be the class of $\lambda \mapsto \tau_\lambda(T)$ in the quotient C^*-algebra $\mathcal{B} := C_b([a, \infty))/C_0([0, \infty))$. One can show that $\dot{\tau}\colon \mathscr{L}^{(1,\infty)}(\mathcal{H})_+ \to \mathcal{B}$ extends to a positive linear map $\dot{\tau}\colon \mathscr{L}^{(1,\infty)}(\mathcal{H}) \to \mathcal{B}$ with the property that for each $S \in B(\mathcal{H})$ and for each $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ it holds $\dot{\tau}(ST) = \dot{\tau}(TS)$.

For every state ω on $\mathcal{B}$, one defines $\mathrm{Tr}_\omega\colon \mathscr{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C}$ by

$$\mathrm{Tr}_\omega(T) := \omega(\dot{\tau}(T))$$

for all $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$, this map Tr_ω then satisfies Eq. (4.3) and Eq. (4.4) in Theorem 4.8. Moreover, we have that $T \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ is measurable if and only if $\lim_{\lambda\to\infty} \tau_\lambda(T)$ exists, in which case

$$⨍ T = \lim_{\lambda\to\infty} \tau_\lambda(T).$$

Note that exhibiting a state ω on the (non-separable) C^*-algebra requires the axiom of choice.

(iii) $\mathcal{M}(\mathcal{H})$ is a vector space and satisfies $STS^{-1} \in \mathcal{M}(\mathcal{H})$ for all invertible $S \in B(\mathcal{H})$ and $T \in \mathcal{M}(\mathcal{H})$. Moreover, one can check that $\mathcal{M}(\mathcal{H}) \subsetneq \mathcal{M}(\mathcal{H})$.

(iv) With the help of *real interpolation theory*, one can construct out of $\mathscr{L}^1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$ a family $\mathscr{L}^{(p,q)}(\mathcal{H})$ with $1 < p < \infty$ and $1 \leq q \leq \infty$ of two-sided ideals in $B(\mathcal{H})$. In fact, $\mathscr{L}^{(p,q)}(\mathcal{H})$ for $q < \infty$ consists of those $T \in K(\mathcal{H})$ which satisfy

$$\sum_{N=1}^{\infty} N^{(\frac{1}{p}-1)q-1} \sigma_N(T)^q < \infty,$$

while $\mathscr{L}^{(p,\infty)}(\mathcal{H})$ consists of those $T \in K(\mathcal{H})$ for which

$$\|T\|_{(p,\infty)} := \sup_{N\in\mathbb{N}} N^{\frac{1}{q}-1}\sigma_N(T) < \infty;$$

it follows that $\mathscr{L}^{(p,\infty)}(\mathcal{H}) = I_{1/p}(\mathcal{H})$ for each $p > 1$. On the diagonal, one finds the *Schatten-ideals* $\mathscr{L}^p(\mathcal{H}) := \mathscr{L}^{(p,p)}(\mathcal{H})$, where the interpolation norm $\|\cdot\|_{(p,p)}$ on $\mathscr{L}^{(p,p)}(\mathscr{H})$ is equivalent to the *Schatten-p-norm*

$$\|T\|_p := \operatorname{Tr}(|T|^p)^{1/p}$$

for $T \in \mathscr{L}^p(\mathcal{H})$. A noncommutative integration theory for which $\mathscr{L}^p(\mathcal{H})$ serves as a analoge of the L^p-space in Lebesgue integration theory, was developed by Segal in the fifties.

Definition 4.11: Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spetrac triple. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is

(i) *p-summable* if $(1 + \mathcal{D}^2)^{-1/2} \in \mathscr{L}^p(\mathcal{H})$,

(ii) (p, ∞)*-summable* if $(1 + \mathcal{D}^2)^{-1/2} \in \mathscr{L}^{(p,\infty)}(\mathcal{H})$,

(iii) θ*-summable* if $e^{-t\mathcal{D}^2} \in \mathscr{L}^1(\mathcal{H})$.

Example 4.12: Consider the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ from Example 1.2, where $\mathcal{A} = C^\infty(\mathbb{T}, \mathbb{C})$, $\mathcal{H} = L^2(\mathbb{T}, m)$ and $\mathcal{D}$ was the closure of

$$\mathcal{D}_0 \colon \mathcal{H} \supset \operatorname{dom}\mathcal{D}_0 \longrightarrow \mathcal{H}, \qquad g \longmapsto \frac{1}{\mathrm{i}}g'$$

with domain $\operatorname{dom}\mathcal{D}_0 := C^1(\mathbb{T})$. Then the operator $\Delta := \mathcal{D}^2$ has spectrum $\operatorname{Sp}(\Delta) = \{|n|^2 \mid n \in \mathbb{Z}\}$. We conclude that $(1 + \Delta)^{-1/2} \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$, i.e., the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $(1, \infty)$-summable. Indeed, since the Fourier transform $\mathcal{F} \colon L^2(\mathbb{T}, m) \to \ell^2(\mathbb{Z})$ is a unitary and $\mathcal{F}\mathcal{D}\mathcal{F}^{-1} = M_{(n)_{n\in\mathbb{Z}}}$, where M_λ, for any sequence $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, is the closed operator

$$M_\lambda \colon \ell^2(\mathbb{Z}) \supseteq \operatorname{dom} M_\lambda \longrightarrow \ell^2(\mathbb{Z}), \qquad (a_n)_{n\in\mathbb{Z}} \longmapsto (\lambda_n a_n)_{n\in\mathbb{Z}}$$

with domain $\operatorname{dom} M_\lambda := \{(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid (\lambda_n a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\}$ we conclude that

$$\mathcal{F}\Delta\mathcal{F}^{-1} = M_{(|n|^2)_{n\in\mathbb{Z}}}, \qquad \mathcal{F}(1 + \Delta)^{-1/2}\mathcal{F}^{-1} = M_{(1+n^2)^{-2})_{n\in\mathbb{Z}}} \in B(\ell^2(\mathbb{Z}))$$

and finally $(1 + \Delta)^{-1/2} \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$; in fact, we have

$$\Big(\mu_n((1 + \Delta)^{-1/2})\Big)_{n\in\mathbb{N}_0} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{1+n^2}}, \frac{1}{\sqrt{1+n^2}}, \dots\right),$$

so that $\gamma_N((1 + \Delta)^{-1/2}) \to 2$ as $N \to \infty$ and hence even $(1 + \Delta)^{-1/2} \in \mathcal{M}(\mathcal{H})$ with $⨍(1 + \Delta)^{-1/2} = 2$.

Example 4.13: In the situation of Example 4.12, let $P \in B(\mathcal{H})$ be the orthogonal projection onto $\ker \mathcal{D} = \mathbb{C}1 \subset L^2(\mathbb{T}, m)$. By Exercise 1B-1, $(\mathcal{A}, \mathcal{H}, \tilde{\mathcal{D}})$ with $\tilde{\mathcal{D}} := \mathcal{D}_P = \mathcal{D} + P$ gives another spectral triple. Note that $\tilde{\Delta} := \tilde{\mathcal{D}}^2 = \Delta + P$, so that $\tilde{\Delta}$ becomes invertible. We have $\tilde{\Delta}^{-1/2} \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ since

$$(\mu_n(\tilde{\Delta}^{-1/2}))_{n \in \mathbb{N}_0} = \left(1, 1, 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n}, \ldots\right)$$

and thus $\tilde{\gamma}_N(\tilde{\Delta}^{-1/2}) \to 2$ as $N \to \infty$; in fact, we have $\tilde{\Delta}^{-1/2} \in \mathcal{M}(\mathcal{H})$ and $⨍ \tilde{\Delta}^{-1/2} = 2$.

More generally, for $f \in \mathcal{A} = C^\infty(\mathbb{T}, \mathbb{C})$, we have $f\tilde{\Delta}^{-1/2} \in \mathscr{L}^{(1,\infty)}(\mathcal{H})$ because $f\cdot\colon L^2(\mathbb{T}, m) \to L^2(\mathbb{T}, m)$, $g \mapsto fg$ is bounded and $\mathscr{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$. It is a consequence of Connes trace theorem that $f \cdot \tilde{\Delta}^{-1/2}$ is measureable and

$$⨍ f \cdot \tilde{\Delta}^{-1/2} = \frac{1}{\pi} \int_{\mathbb{T}} f(\zeta)\, dm(\zeta)\,. \tag{4.5}$$

The general version of Connes trace theorem (1988) is about pseudodifferential operators on compact Riemannian manifolds. This theory has its origins in the work of Kohn, Nirenberg, Hörmander and others in the sixties.

Definition 4.14: Let $\mathcal{M}$ be a compact smooth manifold of dimension n and let $\pi_E\colon E \to \mathcal{M}$ be a k-dimensional smooth vector bundle.

(i) A *differential operator of order* m is a linear operator

$$P\colon \Gamma(\mathcal{M}, E) \longrightarrow \Gamma(\mathcal{M}, E)$$

which, in local coordinates $x = (x_1, \ldots, x_n)$ of $\mathcal{M}$, is of the form

$$P = \sum_{|\alpha| \le m} A_\alpha(x)(-\mathrm{i})^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with entries $0 \le \alpha_1, \ldots, \alpha_n \le n$ and cardinality $|\alpha| := \sum_{j=1}^n \alpha_j$, the $A_\alpha \in M_k(C^\infty(\mathcal{M}))$ for each $|\alpha| \le m$ with $A_\alpha \not\equiv 0$ for some multiindex α with $|\alpha| = n$.

(ii) For $\xi \in T_x^*\mathcal{M}$, written as $\xi = \sum_{j=1}^n \xi_j\, dx_j$, we define the *complete symbol of* P (as the polynomial in $\xi_1, \ldots, \xi_n$ given) by $p^P(x, \xi) := \sum_{d=0}^m p_d^P(x, \xi)$, where

$$p_d^P(x, \xi) := \sum_{|\alpha| = d} A_\alpha(x) \xi^\alpha := \sum_{|\alpha| = d} A_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

The *principal symbol of* P is defined as

$$\sigma^P(x,\xi) := p_m^P(x,\xi) = \sum_{|\alpha|=d} A_\alpha(x)\xi^\alpha.$$

It induces a linear map $\sigma^P(\xi)\colon E_x \to E_x$ for each $\xi \in T_x^*\mathcal{M}$.

(iii) We say that the differential operator P is *elliptic* if its principal symbol $\sigma^P(\xi)\colon E_x \to E_x$ is invertible for each $x \in \mathcal{M}$ and any $\xi \in T_x^*\mathcal{M} - \{0\}$.

(iv) For a local section u of E, one can write

$$(Pu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\mathrm{i}\langle \xi, x\rangle} p^P(x,\xi)\widehat{u}(\xi)\, d\xi_1 \cdots d\xi_n \tag{4.6}$$

with the *Fourier transform* $\widehat{u}$ of u which is given by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}\langle \xi, x\rangle} u(x)\, dx_1 \cdots dx_n\,.$$

A linear operator $P\colon \Gamma(\mathcal{M},E) \to \Gamma(\mathcal{M},E)$ is called *pseudodifferential operator of order* m, written $P \in \Psi^m(\mathcal{M},E)$, if Eq 4.6 holds locally for a matrix-valued function p^P in the *symbol class* $\mathrm{Sym}^m(\mathcal{M},E)$, i.e., in local coordinates, a matrix of smooth functions whose derivatives satisfy the growth conditions

$$|\partial_x^\alpha \partial_\xi^\beta p_{i,j}(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.$$

The *principal symbol of* P is then defined as

$$\sigma^P := [p^P] \in \mathrm{Sym}^m(\mathcal{M},E)/\mathrm{Sym}^{m-1}(\mathcal{M},E).$$

(v) Suppose that g is a Riemannian metric on $\mathcal{M}$. The *Wodzicki residue* of $P \in \Psi^{-n}(\mathcal{M},E)$ is defined by

$$\mathrm{Res}_W(P) := \frac{1}{(2\pi)^n} \int_{S^*\mathcal{M}} \mathrm{tr}\, \sigma^P(x,\xi)\omega_\xi \wedge dx,$$

where $S^*\mathcal{M} := \{(x,\xi) \in T_x^*\mathcal{M} \mid \langle \cdot,\cdot\rangle_{T_x^*\mathcal{M}} = 1\}$ is the *co-sphere bundle over* $\mathcal{M}$, tr is the matrix trace, $dx := dx_1 \wedge \cdots \wedge dx_n$ and

$$\omega_\xi := \sum_{j=1}^n (-1)^{j-1} \xi\, d\xi_1 \wedge \cdots \wedge d\hat{\xi}_j \wedge \cdots \wedge d\xi_n\,.$$

Theorem 4.15 (Connes' trace theorem, 1988): *Let* $(\mathcal{M},g)$ *be a compact Riemannian manifold of dimension* n*. For* $P \in \Psi^{-n}(\mathcal{M},E)$ *with a complex vector bundle* E *over* $\mathcal{M}$*, the following statements hold:*

(i) *P extends to a bounded linear operator on the Hilbert space $L^2(\mathcal{M}, E)$, which is obtained by completion of $\Gamma(\mathcal{M}, E)$ with respect to the inner product given by*

$$\langle u_1, u_2\rangle := \int_{\mathcal{M}} u_2(x)^* u_1(x)\, d\mathrm{vol}(x)\,.$$

Moreover, $P \in \mathscr{L}^{(1,\infty)}(L^2(\mathcal{M}, E)$.

Exercises

Exercise 4.1: Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space.

(i) Let $T \in K(\mathcal{H})$ and $N \in \mathbb{N}$ be given. Prove the formula

$$\sigma_N(T) = \inf\left\{\|R\|_1 + N\|S\| \;\middle|\; R \in \mathscr{L}^1(\mathcal{H}), S \in K(\mathcal{H}) : T = R + S\right\}$$

for the value $\sigma_N(T)$ that was defined in Definition 4.5 of the lecture.

(ii) Like in Remark 4.10 (ii), we define for every $T \in K(\mathcal{H})$ and each $\lambda > 0$

$$\sigma_\lambda(T) := \inf\left\{\|R\|_1 + \lambda\|S\| \;\middle|\; R \in \mathscr{L}^1(\mathcal{H}), S \in K(\mathcal{H})\colon T = R + S\right\}.$$

Due to (i), this interpolates the values $\sigma_N(T)$. Show that this interpolation is in fact piecewise linear, i.e., prove that $\sigma_\lambda(T) = \lambda\|T\|$ holds for every $\lambda \in [0, 1)$ and that

$$\sigma_{N+\lambda}(T) = (1 - \lambda)\sigma_N(T) + \lambda\sigma_{N+1}(T)$$

holds for each $N \in \mathbb{N}$ and every $\lambda \in [0, 1)$.

Exercise 4.2: Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples with infinite dimensional separable complex Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and suppose that the operator $\Gamma_1 \in B(\mathcal{H}_1)$ is a grading on $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$. Put

$$\mathcal{A} := \mathcal{A}_1 \otimes_{\mathbb{C}} \mathcal{A}_2, \qquad \mathcal{H} := \mathcal{H}_1 \mathbin{\overline{\otimes}_{\mathbb{C}}} \mathcal{H}_2, \qquad \text{and} \qquad \mathcal{D} := \mathcal{D}_1 \otimes \mathrm{id}_{\mathcal{H}_2} + \Gamma_1 \otimes \mathcal{D}_2.$$

Prove the following assertions:

(i) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple.

(ii) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable[2] whenever at least one of the spectral triples $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ is θ-summable.

[2] Recall from Definition 4.11 that a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to be *θ-summable* if $e^{-t\mathcal{D}^2} \in \mathscr{L}^1(\mathcal{H})$ for each $t > 0$.

Exercise 4.3: Consider the Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Prove the following properties:

(i) For each $u \in \mathcal{S}(\mathbb{R}^n)$ and each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we have that $\partial^\alpha(\mathcal{F}u) = (-\mathrm{i})^{|\alpha|}\mathcal{F}(m_\alpha u)$, where m_α denotes the function

$$m_\alpha\colon \mathbb{R}^n \longrightarrow \mathbb{C}, \qquad (x_1, \ldots, x_n) \longmapsto x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

(ii) For each $u \in \mathcal{S}(\mathbb{R}^n)$ and each multi-index $\alpha \in \mathbb{N}_0^n$, we have that

$$\mathcal{F}(\partial^\alpha u) = \mathrm{i}^{|\alpha|} m_\alpha \mathcal{F}u.$$

Exercise 4.4: Let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ be open. Consider a differential operator $P\colon C^\infty(\Omega) \to C^\infty(\Omega)$ which is of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha (-i)^{|\alpha|} \partial^\alpha$$

for some integer $m \geq 0$ and with coefficients $a_\alpha \in C^\infty(\Omega)$ for each $|\alpha| \leq m$. Let

$$p^P\colon \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x, \xi) \longmapsto \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$$

be the complete symbol of P. Prove that for each $u \in \mathcal{S}(\mathbb{R}^n)$ and every point $x \in \mathbb{R}^n$

$$(Pu|_\Omega)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\mathrm{i}\langle \xi, x\rangle} p^P(x, \xi) \hat{u}(\xi)\, d\lambda^n(\xi)\,.$$

Appendix

Appendix A.

Basics of unbounded operators

Let $(\mathcal{H}_1, \langle\cdot,\cdot\rangle)$ and $(\mathcal{H}_2, \langle\cdot,\cdot\rangle_2)$ be complex Hilbert spaces; the norms induced by the inner product are denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

I. The notion of unbounded linear operators

By an *unbounded (linear) operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$, we mean a linear map

$$T\colon \mathcal{H}_1 \supseteq \operatorname{dom} T \longrightarrow \mathcal{H}_2$$

that is defined on a linear subspace $\operatorname{dom} T$ of $\mathcal{H}_1$, called the *domain of* T. We say that T is *densely defined* if $\operatorname{dom} T$ is dense in $\mathcal{H}_1$, i.e., if $\mathrm{cl}_{\|\cdot\|_1}(\operatorname{dom} T) = \mathcal{H}_1$.

The *graph of* T, which we will denote by $G(T)$, is defined as

$$G(T) := \{(x, Tx) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid x \in \operatorname{dom} T\}.$$

It is thus linear subspace of the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ with the inner product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle_1 + \langle y_1, y_2 \rangle_2 \qquad \text{for } (x_1, y_1), (x_2, y_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Lemma A.1: *A linear subspace $G \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ is the graph of an unbounded linear operator (i.e., there is an unbounded linear operator $T\colon \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ such that $G = G(T)$) if and only if $G \cap (\{0\} \times \mathcal{H}_2) = \{(0,0)\}$.*

II. Closed and closable operators

Let $T\colon \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ be an unbounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

An unbounded operator $S\colon \mathcal{H}_1 \supseteq \operatorname{dom} S \to \mathcal{H}_2$ is called an *extension of* T, written as $S \subseteq T$, if $G(T) \subseteq G(S)$ holds, i.e., if $\operatorname{dom} T \subseteq \operatorname{dom} S$ and $Sx = Tx$ for every $x \in \operatorname{dom} T$.

We say that the unbounded operator T is

- *closed*, if $G(T)$ is closed in $\mathcal{H}_1 \oplus \mathcal{H}_2$; explicitly, this means that for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} T$ which converges in $\mathcal{H}_1$ to a point $x \in \mathcal{H}_1$ and for which $(Tx_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{H}_2$ to a point $y \in \mathcal{H}_2$, it holds true that $x \in \operatorname{dom} T$ and $y = Tx$.
- *closable*, if T admits an extension S that is closed.

Theorem A.2: *For an unbounded operator $T : \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$, the following statements are equivalent:*

(i) *T is closable;*

(ii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} T$ which converges to 0 in $\mathcal{H}_1$ and for which $(Tx_n)_{n\in\mathbb{N}}$ converges in $\mathcal{H}_2$ to a point $y \in \mathcal{H}_2$, we necessarily have that $y = 0$;*

(iii) $\operatorname{cl}(G(T)) \cap (\{0\} \times \mathcal{H}_2) = \{(0,0)\}$.

It is worthwhile to take a closer look on the proof that (iii) implies (i). It follows from Lemma A.1 that if (iii) holds, then $\operatorname{cl}(G(T))$ must be the graph of an unbounded linear operator, say $\overline{T}\colon \mathcal{H}_1 \supseteq \operatorname{dom} \overline{T} \to \mathcal{H}_2$. The operator $\overline{T}$ is thus a closed extension of T; in fact, it is the (unique) minimal closed extension (i.e., for every other closed operator S that satisfies $T \subseteq S$, it follows that $\overline{T} \subseteq S$), called the *closure of* T. Furthermore, its domain $\operatorname{dom} \overline{T}$ is the closure of $\operatorname{dom} T$ with respect to the *graph norm* $\|\cdot\|_T$ which is defined by $\|x\|_T^2 := \|x\|_1^2 + \|Tx\|_2^2$ for each $x \in \operatorname{dom} T$.

III. The adjoint operator

Let now $T\colon \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ be a densely defined unbounded linear operator. For every $y \in \mathcal{H}_2$, we introduce a linear functional

$$\phi_y\colon \operatorname{dom} T \longrightarrow \mathbb{C}, \qquad x \longmapsto \langle Tx, y\rangle_2.$$

Using this notation, we may define

$$\operatorname{dom} T^* := \{y \in \mathcal{H}_2 \mid \phi_y \text{ is continuous on } \operatorname{dom} T \text{ with respect to } \|\cdot\|_1\},$$

which is clearly a subspace of $\mathcal{H}_2$. Since $\operatorname{dom} T$ is dense in $\mathcal{H}_1$, ϕ_y for every $y \in \operatorname{dom} T^*$ extends uniquely to a bounded linear functional Φ_y on $\mathcal{H}_1$; by the Riesz representation theorem, the latter must be of the form $\Phi_y(x) = \langle x, T^*y\rangle_1$ for all $x \in \mathcal{H}_1$ with a unique vector $T^*y \in \mathcal{H}_1$. The assignment $y \mapsto T^*y$ is in fact linear on $\operatorname{dom} T^*$, so that this construction results in an unbounded linear operator

$$T^*\colon \mathcal{H}_2 \supseteq \operatorname{dom} T^* \longrightarrow \mathcal{H}_1,$$

called the *adjoint of* T.

Theorem A.3: *Let* $T\colon \mathcal{H}_1 \supseteq \operatorname{dom} T \to \mathcal{H}_2$ *be a densely defined unbounded linear operator.*

(i) *The adjoint operator* T^* *is always closed.*

(ii) *If* T *is closed, then* T^* *is densely defined and the operator* $T^{**} := (T^*)^*$ *satisfies* $T^{**} = T$.

(iii) *The operator* T *is closable if and only if its adjoint* T^* *is densely defined; in this case, we have that* $T^{**} = \overline{T}$.

IV. Symmetric and selfadjoint operators

Throughout the following, let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. A densely defined operator $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ is called

- *symmetric*, if $T \subseteq T^*$, or in other words, if $\langle Tx_1, x_2\rangle = \langle x_1, Tx_2\rangle$ for all $x_1, x_2 \in \mathcal{H}$.
- *selfadjoint*, if $T = T^*$.
- *maximally symmetric*, if T is symmetric and if the following holds: whenever S is a symmetric extension of T, it follows that $S = T$.
- *essentially selfadjoint*, if T is symmetric with selfadjoint closure $\overline{T}$.

Lemma A.4:

(i) *Every symmetric operator is closable.*

(ii) *Every selfadjont operator is maximally symmetric.*

(iii) *A densely defined operator* $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ *is essentially selfadjoint if and only if* $\overline{T} = T^*$.

Suppose that T is a densely defined and symmetric operator. The *defect indices* $n_\pm(T) \in [0,\infty]$ of T are defined by

$$n_+(T) := \dim(T+i)^\perp = \dim\ker(T^* - i) \quad \text{and} \quad n_-(T) := \dim(T-i)^\perp = \dim\ker(T^* + i).$$

Theorem A.5: *Let $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be densely defined and symmetric. Then the following statements are equivalent:*

(i) *T is essentially selfadjoint;*

(ii) *$n_+(T) = n_-(T) = 0$;*

(iii) *$\operatorname{ran}(T+i)$ and $\operatorname{ran}(T-i)$ are dense.*

Suppose in addition that T is closed. Then the following statements are equivalent:

(i) *T is selfadjoint;*

(ii) *$n_+(T) = n_-(T) = 0$;*

(iii) *$\operatorname{ran}(T+i) = \operatorname{ran}(T-i) = \mathcal{H}$.*

For closed operators, we actually have the following.

Theorem A.6: *Let $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be densely defined, closed, and symmetric. Then we have the following:*

(i) *T is selfadjoint if and only if $n_+(T) = n_-(T) = 0$.*

(ii) *T is maximally symmetric if and only if $n_+(T) = 0$ or $n_-(T) = 0$.*

(iii) *T has a selfadjoint extension if and only if $n_+(T) = n_-(T)$.*

V. Resolvent set and spectrum

For any densely defined unbounded linear operator $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$, we define its *resolvent set* $\rho(T)$ by

$$\rho(T) := \left\{ \lambda \in \mathbb{C} \;\middle|\; (T - \lambda 1)\colon \operatorname{dom} T \to \mathcal{H} \text{ is bijective and } (T-\lambda 1)^{-1} \in B(\mathcal{H}) \right\}$$

and its *spectrum* $\sigma(T)$ by $\sigma(T) := \mathbb{C} - \rho(T)$.

Lemma A.7: *Suppose that $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ is densely defined and closed.*

(i) *If* $(T - \lambda 1)\colon \operatorname{dom} T \to \mathcal{H}$ *is bijective for a* $\lambda \in \mathbb{C}$*, then its inverse* $(T - \lambda 1)^{-1}$ *is bounded.*

(ii) *The spectrum* $\sigma(T) \subseteq \mathbb{C}$ *is closed.*

(iii) *If* T *is selfadjoint, then* $\sigma(T) \subseteq \mathbb{R}$*.*

(iv) *If* T *is symmetric and satisfies* $\sigma(T) \subseteq \mathbb{R}$*, then* T *is selfadjoint.*

VI. The spectral theorem and functional calculus

Theorem A.8: *Let* $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ *be selfadjoint. Then there is a unique spectral measure* E *such that*

$$\langle Tx, y\rangle = \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)x, y\rangle \qquad \text{for all } x \in \operatorname{dom} T,\ y \in \mathcal{H}.$$

If $h\colon \mathbb{R} \to \mathbb{R}$ *is measurable, then*

$$\langle h(T)x, y\rangle = \int_{\mathbb{R}} h(\lambda)\, d\langle E(\lambda)x, y\rangle$$

defines a selfadjoint operator $h(T)\colon \mathcal{H} \supseteq \operatorname{dom} h(T) \to \mathcal{H}$ *with domain*

$$\operatorname{dom} h(T) := \left\{ x \in \mathcal{H} \;\middle|\; \int_{\mathbb{R}} |h(\lambda)|^2 \, d\langle E(\lambda)x, x\rangle < \infty \right\}.$$

Appendix B.

Basics on Fourier transform

We recall some basic facts about the Fourier transform on $\mathbb{R}^n$ which can be used for the solution of the exercises without proof.

(i) Let λ^n be the Lebesgue measure on $\mathbb{R}^n$. For every function $u \in L^1(\mathbb{R}^n, \lambda^n)$, we define its *Fourier transform* $\hat{u} = \mathcal{F}u$ by

$$(\mathcal{F}u)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}\langle \xi, x\rangle} u(x)\, d\lambda^n(x) \qquad \text{for each } \xi \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, i.e., $\langle \xi, x\rangle = \sum_{j=1}^n \xi_j x_j$ for each $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ in $\mathbb{R}^n$. It is known that $\mathcal{F}f \in C_0(\mathbb{R}^n)$.

(ii) Let $\mathcal{S}(\mathbb{R}^n)$ be the *Schwartz space*, i.e., the space of all smooth functions $f\colon \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^m)|(\partial^\alpha f)(x)| < \infty$$

for each $m \in \mathbb{N}_0$ and each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. The Fourier transform $\mathcal{F}$, if restricted to $\mathcal{S}(\mathbb{R}^n)$, has the remarkable property that it induces a bijection $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, with inverse given by

$$(\mathcal{F}^{-1}v)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\mathrm{i}\langle \xi, x\rangle} v(\xi)\, d\lambda^n(\xi) \qquad \text{for each } x \in \mathbb{R}^n.$$

Appendix C.

Solutions to the exercises

Solution (to Exercise 1.1): According to Definition 1.1, we have to check that

(i) $\mathcal{D}_V$ is selfadjoint,

(ii) $\mathcal{D}_V$ has compact resolvent,

(iii) For all $a \in \mathcal{A}$, $\pi(a) \operatorname{dom} \mathcal{D}_V$ is contained in $\mathcal{D}_V$ and $[\mathcal{D}_V, \pi(a)]$ is bounded on $\mathcal{H}$.

Ad (i): This is a general fact: If $T \colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ and $S \in B(\mathcal{H})$, $(T+S)^* = T^* + S^*$. In particular, if T and S are selfadjoint, then $T + S$ is also selfadjoint.

To see this, note that

$$\begin{aligned}\operatorname{dom}(T+S)^* &= \{y \in \mathcal{H} \mid x \mapsto \langle (T+S)x, y\rangle = \langle Tx, y\rangle + \langle Sx, y\rangle \text{ is bounded}\} \\ &= \{y \in \mathcal{H} \mid x \mapsto \langle Tx, y\rangle \text{ is bounded}\} \\ &= \operatorname{dom} T^*\end{aligned}$$

since $x \mapsto \langle Sx, y\rangle$ is bounded for every $y \in \mathcal{H}$. Thus, for all $x \in \operatorname{dom} T$ and for all $y \in \operatorname{dom} T^*$ it holds

$$\langle Tx, y\rangle + \langle Sx, y\rangle = \langle (T+S)x, y\rangle = \langle x, (T+S)^* y\rangle = \langle x, (T^* + S^*)y\rangle,$$

i.e., $(T+S)^* = T^* + S^*$.

Ad (ii): Take any $\lambda \in \mathbb{C} - \sigma(\mathcal{D}_V)$, choose $\lambda_1 \in \mathbb{C} - \sigma(\mathcal{D})$ and $\lambda_2 := \lambda - \lambda_1$. Then

$$\begin{aligned}&(\mathcal{D}_V - \lambda 1)^{-1} - (\mathcal{D} - \lambda_1 1)^{-1} \\ &\qquad = (\mathcal{D} - \lambda_1 1)^{-1}\big((\mathcal{D} - \lambda_1 1) - (\mathcal{D}_V - \lambda 1)\big)(\mathcal{D}_V - \lambda 1)^{-1} \\ &\qquad = -(\mathcal{D} - \lambda_1 1)^{-1}(V - \lambda_2 1)(\mathcal{D}_V - \lambda 1)^{-1},\end{aligned}$$

i.e., we have

$$(\mathcal{D}_V - \lambda 1)^{-1} = (\mathcal{D} - \lambda_1 1)^{-1}(1 - (V - \lambda_2 1)(\mathcal{D}_V - \lambda 1)^{-1}),$$

where $(\mathcal{D} - \lambda_1 1)^{-1}$ is compact and $(1 - (V - \lambda_2 1)(\mathcal{D}_V - \lambda 1)^{-1})$ is bounded, which yields that $(\mathcal{D}_V - \lambda 1)^{-1}$ is compact (recall that $K(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$).

Ad (iii): For all $a \in \mathcal{A}$, $\pi(a) \operatorname{dom} \mathcal{D}_V \subseteq \operatorname{dom} \mathcal{D}_V$, as $\operatorname{dom} \mathcal{D}_V = \operatorname{dom} \mathcal{D}$. Furthermore,

$$[\mathcal{D}_V, \pi(a)] = [\mathcal{D}, \pi(a)] + [V, \pi(a)],$$

where $[V, \pi(a)] \in B(\mathcal{H})$, extends to a bounded operator on $\mathcal{H}$.

Solution (to Exercise 2.1): (i) First we want to show, that

$$\{\partial_j|_{x_0} \mid 1 \leq j \leq n\}$$

is $\mathbb{R}$-linearly independent. Let therefore $\alpha^1, \ldots, \alpha^n \in \mathbb{R}$ with $\sum_{j=1}^n \alpha^j \partial_j|_{x_0} = 0$ in $T_{x_0}\mathbb{R}^n$. Then, for each $[f]_{x_0} \in C^\infty_{x_0}(\mathbb{R})$, it holds

$$0 = \sum_{j=1}^n \alpha^j \partial_j|_{x_0}([f]_{x_0}) = \sum_{j=1}^n \alpha^j \frac{\partial f}{\partial x^j}(x_0).$$

For $1 \leq i \leq n$, we apply this to the equivalence classes $[f_i]$ of the functions

$$f_i \colon \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x^1, \ldots, x^n) \longmapsto x^i - x^i_0,$$

which gives $0 = \sum_{j=1}^n \alpha^j \frac{\partial f_i}{\partial x^j}(x_0) = \alpha^i$, as desired.

For $\{\partial_j|_{x_0} \mid 1 \leq j \leq n\}$ to be a basis of $T_{x_0}\mathbb{R}^n$, we need to show that indeed $\operatorname{Lin}\{\partial_j|_{x_0} \mid 1 \leq j \leq n\} = T_{x_0}\mathbb{R}^n$. To see this, we first need a technical result. For every open set $U \subseteq \mathbb{R}^n$ and each smooth function $f \colon U \to \mathbb{R}$, we find by Taylor's theorem on every open ball $B(x_0, r) \subseteq U$ with $r > 0$ a smooth function $\varphi \colon B(x_0, r) \to \mathbb{R}$ such that for all $x \in B(x_0, r)$ we have

$$f(x) = f(x_0) + \langle (\operatorname{grad} f)(x_0), x - x_0 \rangle + \varphi(x),$$

where $\varphi(x)/(x - x_0) \to 0$ for $x \to x_0$. In fact, we can give such a function φ explicitly: For all $x \in B(x_0, r)$ we put

$$\varphi(x) := \langle g(x), x - x_0 \rangle = \sum_{j=1}^n g^j(x) f_i(x)$$

where $g = (g^1, \dots, g^n) \colon B(x_0, r) \to \mathbb{R}^n$ is given by

$$g^j(x) := \int_0^1 \left[\frac{\partial f}{\partial x^j}(x_0 + s(x - x_0)) - \frac{\partial f}{\partial x^j}(x_0) \right] ds\,.$$

Note that each g^j, $1 \le j \le n$, is smooth with $g^j(x_0) = 0$.

For any $\delta \in T_{x_0}\mathbb{R}^n$, any representative f of the class $[f]_{x_0}$ in $C^\infty_{x_0}(\mathbb{R})$ and the corresponding smooth function φ defined as above we thus find

$$\delta([\varphi]_{x_0}) = \delta\Big(\sum_{j=1}^n [g^j]_{x_0} [f_j]_{x_0} \Big) = \sum_{j=1}^n \Big(\delta([g^j]_{x_0}) f_j(x_0) + g^j(x_0) \delta([f_j]_{x_0}) \Big) = 0$$

and hence

$$\delta([f]_{x_0}) = \sum_{j=1}^n \delta([f_j]_{x_0}) \frac{\partial f}{\partial x^j}(x_0) \in \mathrm{Lin}\{\partial_j|_{x_0} \mid 1 \le j \le n\},$$

since $\delta([f_j]_{x_0}) \in \mathbb{R}$ for $1 \le j \le n$, as we wanted to show.

(ii) Note that we have an isomorphism

$$\phi_{i,x_0} \colon C^\infty_{x_0}(\mathcal{M}) \longrightarrow C^\infty_{\varphi_i(x_0)}(\mathbb{R}^n), \qquad [f]_{x_0} \longmapsto [f \circ \varphi_i^{-1}]_{\varphi_i(x_0)}$$

and thus an isomorphism

$$\widehat{\phi}_{i,x_0} \colon T_{\varphi_i(x_0)}\mathbb{R}^n \longrightarrow T_{x_0}\mathcal{M}, \qquad \delta \longmapsto \delta \circ \Phi_{i,x_0}.$$

Take the isomorphism from (i), i.e., the map

$$\psi_{\varphi_i(x_0)} \colon \mathbb{R}^n \longrightarrow T_{\varphi_i(x_0)}\mathbb{R}^n, \qquad (v^1, \dots, v^n) \longmapsto \sum_{j=1}^n v^j \partial_j|_{\varphi_i(x_0)},$$

this yields an isomorphism $\Theta_{i,x_0} := \widehat{\phi}_{i,x_0} \circ \psi_{\varphi_i(x_0)} \colon \mathbb{R}^n \to T_{x_0}\mathcal{M}$, which at $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ and $[f]_{x_0} \in C^\infty_{x_0}(\mathcal{M})$ looks as

$$\begin{aligned}
\Theta_{i,x_0}(v)([f]_{x_0}) &= \widehat{\phi}_{i,x_0}\Big(\sum_{j=1}^n v^j \partial_j|_{\varphi_i(x_0)} \Big)([f]_{x_0}) \\
&= \Big(\sum_{j=1}^n v^j \partial_j|_{\varphi_i(x_0)} \Big)(\phi_{i,x_0}([f]_{x_0})) \\
&= \Big(\sum_{j=1}^n v^j \partial_j|_{\varphi_i(x_0)} \Big)([f \circ \varphi_i^{-1}]_{\varphi_i(x_0)}) = \sum_{j=1}^n v^j (\partial_j(f \circ \varphi_i^{-1}))(\varphi_i(x_0))
\end{aligned}$$

which is the given assertion.

Solution: (i) We define

$$E^* := \coprod_{x\in X} E_x^* = \{(x,f) \mid x \in X, f \in E_x^*\}, \qquad \pi^* \colon E^* \longrightarrow X, (x,f) \longmapsto x.$$

Let $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ be a maximal bundle atlas for the bundle $\pi\colon E \to X$, with local trivialisation $\tau_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{K}^n$. By definition, for each $x \in U_i$ the map $\tau_i|_{E_x}\colon E_x \to \{x\} \times \mathbb{K}^n$ is an isomorphism, thus

$$(\tau_i|_{E_x})'\colon (K^n)^* \longrightarrow E_x^*, \qquad g \longmapsto g \circ (\tau_i|_{E_x})$$

is an isomorphism as well. Fixing an isomorphism $\Phi\colon \mathbb{K}^n \to (\mathbb{K}^n)^*$ allows us to define $\tau_i^*\colon (\pi^*)^{-1}(U_i) \to U_i \times \mathbb{K}^n$ fibre-wise by

$$\tau_i^*|_{E_x^*} := \Big((\tau_i|_{E_x})' \circ \Phi\Big)^{-1}\colon E_x^* \longrightarrow \mathbb{K}^n = \{x\} \times \mathbb{K}^n.$$

Endowing E^* with the topology, whose open sets are characterised by "$W \subseteq E^*$ is open if and only if $\tau_i^*(U_i \cap W) \subseteq U_i \times \mathbb{K}^n$ is open for every $i \in I$" makes E^* a vector bundle with bundle atlas $\mathcal{A}^* = \{(U_i, \tau_i^*) \mid i \in I\}$.

(ii) The transition maps $\sigma_{i,j}^*$ of E^* are of the form

$$\sigma_{i,j}^*(x,v) = (x, S_{i,j}^*(x)v) \qquad \forall\, (x,v) \in (U_i \cap U_j) \times \mathbb{K}^n$$

with the transition matrices $S_{i,j}^*\colon U_i \cap U_j \to \mathrm{Gl}_n(\mathbb{K})$ that are determined at any point $x \in U_i \cap U_j$ by

$$\begin{aligned} L_{\mathcal{E}}^{\mathcal{E}}(S_{i,j}^*(x)) &= \tau_j^*|_{E_x^*} \circ (\tau_i^*|_{E_x^*})^{-1} \\ &= ((\tau_j|_{E_x})' \circ \Phi)^{-1} \circ ((\tau_i|_{E_x})' \circ \Phi) \\ &= \Phi^{-1} \circ ((\tau_i|_{E_x}) \circ (\tau_j|_{E_x})^{-1})' \circ \Phi \\ &= \Phi^{-1} \circ (L_{\mathcal{E}}^{\mathcal{E}}(S_{j,i}(x)))' \circ L_{\mathcal{E}^*}^{\mathcal{E}}(I_n) \\ &= \Phi^{-1} \circ L_{\mathcal{E}^*}^{\mathcal{E}^*}(S_{j,i}(x)^\top) \circ \Phi = L_{\mathcal{E}}^{\mathcal{E}}(S_{j,i}(x)^\top) \end{aligned}$$

where $S_{i,j}\colon U_i \cap U_j \to \mathrm{Gl}_n(\mathbb{K})$ are the transition matrices for E, $\mathcal{E}$ is the standard basis of $\mathbb{K}^n$, $\mathcal{E}^*$ is the dual basis to $\mathcal{E}$ and we fix Φ that satisfies $\Phi(e_i) = e_i^*$ for $i \in \{1, \dots, n\}$.

From the above calculation we conclude $S_{i,j}^*(x) = S_{j,i}(x)^\top = (S_{i,j}(x)^{-1})^\top$. In particular, if E is smooth, then E^* is smooth as well.