2. Spectral triples associated to manifolds

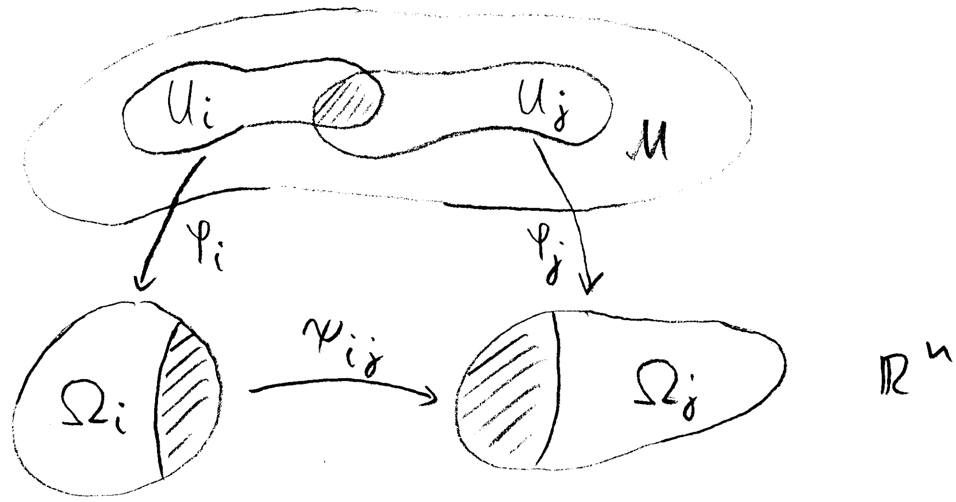
Spectral triples are (supposed to be) the right framework to extend classical differential geometry to the noncommutative world. It is however not clear offhand, how usual manifolds fit into that frame. In this chapter, we will see that indeed each compact oriented smooth manifold induces a commutative spectral triple in a natural way.

2.1. Definition (manifolds):

(i) An n-dimensional topological manifold is a Hausdorff topological space M which is locally euclidean, i.e., each point $x \in M$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

(ii) A (local) chart (U, φ) of M consists of an open set $U \subseteq M$ and a homeomorphism $\varphi : U \to \Omega = \varphi(U)$ onto an open subset Ω of $\mathbb{R}^n$.

(iii) A family $A = \{(U_i, \varphi_i) \mid i \in I\}$ of charts satisfying $M = \bigcup_{i \in I} U_i$ is called atlas of M.
The homeomorphisms $\psi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ given by $\psi_{ij} := \varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(U_i \cap U_j)}$ are called transition maps.



(iv) An atlas $\mathcal{A}$ is called <u>smooth</u> if all its transition maps are smooth (i.e., C^∞).

A chart (U, φ) is said to be <u>smooth with respect to a smooth atlas $\mathcal{A}$</u> if $\mathcal{A} \cup \{(U, \varphi)\}$ is again a smooth atlas.

A smooth atlas $\mathcal{A}$ is called <u>maximal</u> if every chart (U, φ) that is smooth with respect to $\mathcal{A}$ already belongs to $\mathcal{A}$.

Every smooth atlas $\mathcal{A}$ induces a maximal one by

$$\mathcal{A}_{max} := \{(U, \varphi) \mid (U, \varphi) \text{ chart, smooth w.r.t } \mathcal{A}\}.$$

(v) An <u>n-dimensional smooth manifold</u> is an n-dimensional topological manifold M with a maximal smooth atlas $\mathcal{A}$.

<u>2.2 Definition</u> (tangent space):

Let M be an n-dimensional smooth manifold. Fix $x_0 \in M$.

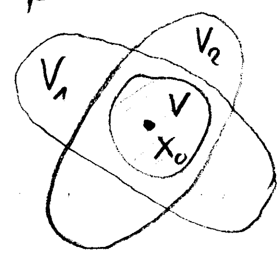
(i) A function $f: V \to \mathbb{R}$ on an open subset $V \subseteq M$ is said to be <u>smooth</u> if

$$f \circ \varphi^{-1}\big|_{\varphi(U \cap V)} : \varphi(U \cap V) \longrightarrow \mathbb{R}$$

is smooth for every smooth chart (U, φ).

(ii) On the set of all pairs (V, f) consisting of

- an open neighborhood V of x_0 and
- a smooth function $f: V \to \mathbb{R}$,



we introduce an equivalence relation $\sim$ by

$$(V_1, f_1) \sim (V_2, f_2) :\Longleftrightarrow \exists\, V \subseteq V_1 \cap V_2 \text{ open},\ x_0 \in V : \quad f_1|_V = f_2|_V$$

The equivalence class of (V, f), denoted by $[f]_{x_0}$, is called the <u>germ</u> of f at x_0.

We denote by $C^\infty_{x_0}(M)$ the $\mathbb{R}$-algebra of germs at x_0

(iii) The $\mathbb{R}$-vector space $T_{x_0}M$ of all linear maps $\delta: C^\infty_{x_0}(M) \to \mathbb{R}$ satisfying the product rule

$$\delta\big([f]_{x_0} \cdot [g]_{x_0}\big) = \delta([f]_{x_0})\, g(x_0) + f(x_0)\, \delta([g]_{x_0})$$

for all $[f]_{x_0}, [g]_{x_0} \in C^\infty_{x_0}(M)$ is called the <u>tangent space</u> to M at x_0.

2.3. Remark:

In the situation of Definition 2.2, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth path, i.e. γ is continuous and

$$\varphi \circ \gamma|_{\gamma^{-1}(U)} : \gamma^{-1}(U) \to \mathbb{R}^n$$

is smooth for every smooth chart (U, φ), such that $\gamma(0) = x_0$. We call $\gamma'(0) \in T_{x_0}M$ given by

$$\gamma'(0)([f]_{x_0}) := (f \circ \gamma)'(0) \qquad \forall [f]_{x_0} \in C^\infty_{x_0}(M)$$

the velocity vector of γ at x_0.

For every $\delta \in T_{x_0}M$, there exists a smooth path $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x_0$ and $\gamma'(0) = \delta$.

Next, we "glue" the tangent spaces, yielding the so-called tangent bundle.

2.4. Definition (topological vector bundle)

Let X be a Hausdorff topological space.

(i) An n-dimensional (real/complex) vector bundle over X is given by a topological space E and a continuous map $\pi : E \to X$ such that

- the fibre $E_x := \pi^{-1}(\{x\})$ is a (real/complex) vector space of dimension n for each $x \in X$,

- for each $x_0 \in X$, there is an open neighborhood U of x_0 and a homeomorphism

$$\tau : \pi^{-1}(U) \to U \times \mathbb{K}^n \qquad (\mathbb{K} = \mathbb{R}, \mathbb{C})$$

such that $\pi|_{\pi^{-1}(U)} = pr_U \circ \tau$, where $pr_U : U \times \mathbb{K}^n \to U$ is the projection on the first component, i.e., $pr_U(x, v) = x$, and $\tau|_{E_x} : E_x \to \{x\} \times \mathbb{K}^n = \mathbb{K}^n$ is a vector space isomorphism for all $x \in U$.

We call (U, τ) a <u>bundle chart</u> (or <u>local trivialization</u>).

(ii) A family $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ of bundle charts satisfying $X = \bigcup_{i \in I} U_i$ is called <u>bundle atlas</u>.

The <u>transition maps</u> $\sigma_{ij} : (U_i \cap U_j) \times \mathbb{K}^n \to (U_i \cap U_j) \times \mathbb{K}^n$ given by $\sigma_{ij} := \tau_j \circ \tau_i^{-1}|_{(U_i \cap U_j) \times \mathbb{K}^n}$ satisfy

$$\begin{tikzcd}
 & \pi^{-1}(U_i \cap U_j) \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & \\
(U_i \cap U_j) \times \mathbb{K}^n \arrow[rr, "\sigma_{ij}"] & & (U_i \cap U_j) \times \mathbb{K}^n \\
(x, v) \arrow[rr, mapsto] & & \sigma_{ij}(x, v) = (x, S_{ij}(x) v)
\end{tikzcd}$$

for a continuous map $S_{ij} : U_i \cap U_j \to GL_n(\mathbb{K})$, called the <u>transition matrices</u>.

<u>2.5. Definition</u> (smooth vector bundle):

Let M be a smooth manifold. An <u>n-dimensional smooth vector bundle over M</u> is an n-dimensional topological vector bundle over $\mathcal{M}$ with a (maximal) bundle atlas $\mathcal{A}$ for which all transition maps are smooth.

2.6. Definition (tangent bundle):

Let $\mathcal{M}$ be an n-dimensional smooth manifold with maximal smooth atlas $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$. We put

$$T\mathcal{M} := \coprod_{x \in \mathcal{M}} T_x\mathcal{M} = \{(x, \delta) \mid x \in \mathcal{M}, \delta \in T_x\mathcal{M}\}$$

and define $\pi : T\mathcal{M} \to \mathcal{M}$ by $\pi(x,\delta) := x$; often, we identify (x,δ) and δ.

We define (candidates for) local trivializations by

$$\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n, \quad (\underset{\substack{\cap \\ T_x\mathcal{M}}}{x, \delta}) \mapsto (x, \Theta_{i,x}^{-1}(\delta))$$

with the isomorphism $\Theta_{i,x} : \mathbb{R}^n \to T_x\mathcal{M}$ given by

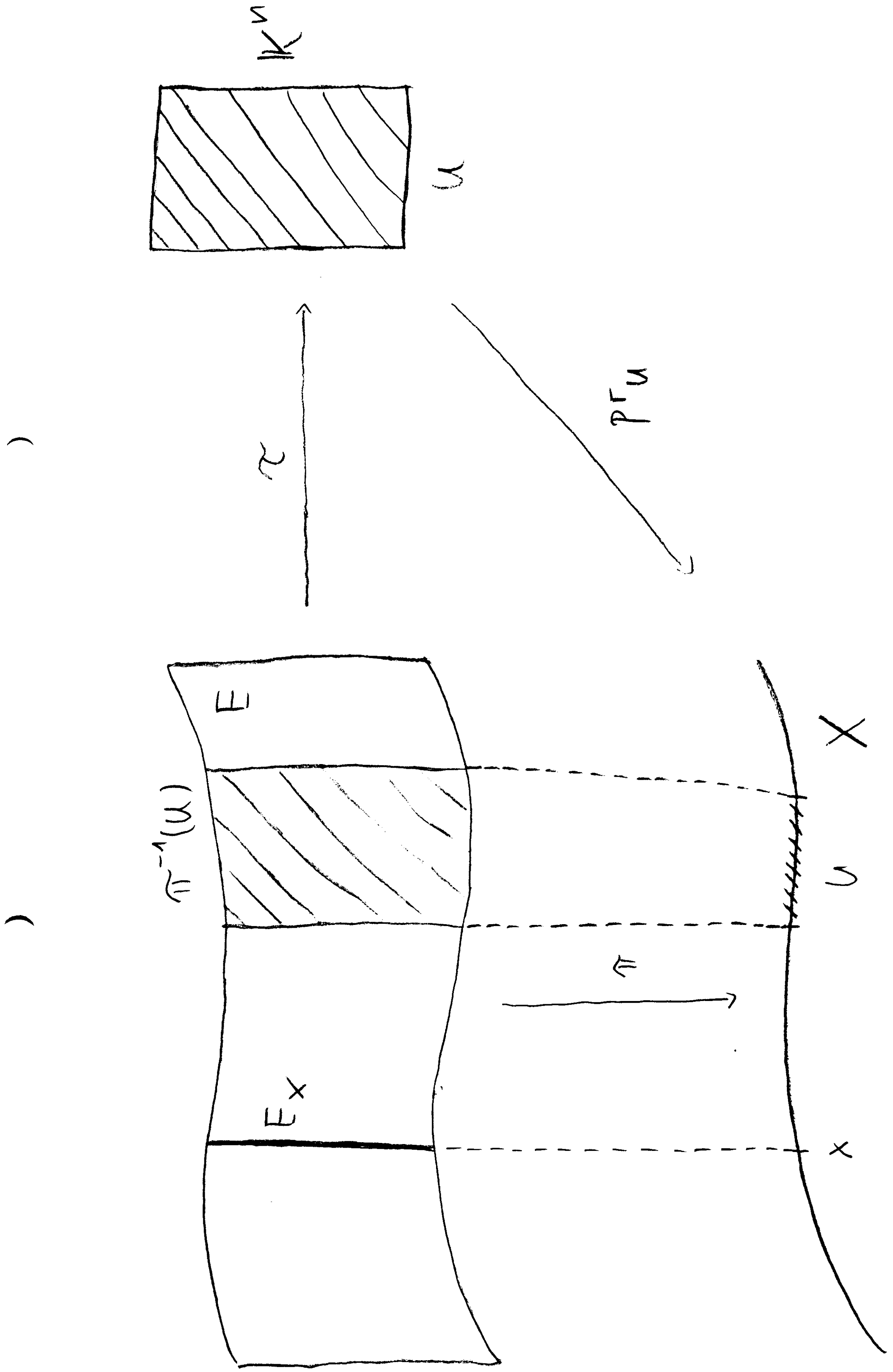
$$\Theta_{i,x}(v)([f]_x) := \sum_{j=1}^{n} v_j \, \partial_j (f \circ \varphi_i^{-1})(\varphi_i(x))$$

for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $[f]_x \in C_x^\infty(\mathcal{M})$.

We endow $T\mathcal{M}$ the topology defined by

$$W \subseteq T\mathcal{M} \text{ open } :\Longleftrightarrow \forall i \in I : \tau_i(U_i \cap W) \subseteq U_i \times \mathbb{R}^n \text{ open}.$$

Then $T\mathcal{M}$ is a n-dimensional smooth vector bundle over $\mathcal{M}$, called the <u>tangent bundle of $\mathcal{M}$</u>.


K^n
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x

<u>2.7. Definition</u> (smooth sections):

Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M and let $V \subseteq M$ be open.
A map $s : V \to E$ is called a <u>smooth section</u> if

(i) $\pi \circ s = \mathrm{id}_V$,

(ii) for each local trivialization $\tau : \pi^{-1}(U) \to U \times \mathbb{K}^n$, we have a smooth composition

$$\tau \circ s|_{U \cap V} : U \cap V \to (U \cap V) \times \mathbb{K}^n$$

In fact, $(\tau \circ s)(x) = (x, f(x))$ for all $x \in U \cap V$ with $f : U \cap V \to \mathbb{K}^n$ being a smooth function.

We write $\Gamma^\infty(V, E)$ for the set of all smooth sections $s : V \to E$, which is a $\mathbb{K}$-vector space.

<u>2.8. Definition</u> (vector fields):

Let M be a smooth manifold. A <u>vector field on M</u> is a smooth section of the tangent bundle TM.

We write $\mathfrak{X}(V) := \Gamma^\infty(V, TM)$ for $V \subseteq M$ open.

<u>2.9. Theorem:</u>

Let M be a smooth manifold of dimension n and let $V \subseteq M$ be open. Then

$$\Phi : \mathfrak{X}(V) \to \operatorname{der} C^\infty(V), \quad (\Phi(X)f)(x) = X(x)([f]_x),$$

is an isomorphism of $\mathbb{R}$-vector spaces