Note that $\mathrm{der}\, C^\infty(V)$ denotes the space of <u>derivations</u> on $C^\infty(V)$, i.e., linear maps $D: C^\infty(V) \to C^\infty(V)$ that satisfy the product rule

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \quad \forall f, g \in C^\infty(V).$$

<u>Proof</u>:

① Take $X \in \mathfrak{X}(V)$ and $f \in C^\infty(V)$.

<u>Claim</u>: $h: V \to \mathbb{R}$, $x \mapsto \underbrace{X(x)}_{\in T_x M}([f]_x)$ is smooth.

<u>Proof</u>: Consider the bundle atlas $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ that we introduced in Definition 2.6. It suffices to show that $h|_{U_i \cap V}$ is smooth for all $i \in I$.

Fix $i \in I$. By Definition 2.7 (ii), we find a smooth function $g = (g_1, \dots, g_n): U_i \cap V \to \mathbb{R}^n$ such that $\tau_i(X(x)) = (x, g(x))$ for all $x \in U_i \cap V$.

Thus $X(x) = \tau_i^{-1}(x, g(x)) = (x, \Theta_{i,x}(g(x)))$ and

$$h(x) = X(x)([f]_x) = \Theta_{i,x}(g(x))([f]_x)$$

$$= \sum_{j=1}^{n} g_j(x) \left(\partial_j (f \circ \varphi_i^{-1})\right)(\varphi_i(x)) \quad \forall x \in U_i \cap V,$$

which shows that h is smooth on $U_i \cap V$. $\square$

② Take $X \in \mathfrak{X}(V)$. By ①, we have a well-defined map

$$\Phi(X) : C^\infty(V) \to C^\infty(V), \ (\Phi(X)f)(x) = X(x)([f]_x).$$

<u>Claim</u>: $\Phi(X)$ is a derivation.

<u>Proof</u>: Let $f, g \in C^\infty(V)$ be given. Then, for all $x \in V$,

$$\begin{aligned}(\Phi(X)(f\cdot g))(x) &= X(x)([f]_x \cdot [g]_x) \\ &= X(x)([f]_x)\cdot g(x) + f(x)\cdot X(x)([g]_x) \\ &= (\Phi(X)(f)\cdot g + f\cdot \Phi(X)(g))(x). \end{aligned}$$ $\square$

③ By ②, we have a well-defined map

$$\Phi : \mathfrak{X}(V) \to \operatorname{der} C^\infty(V), \ X \mapsto \Phi(X),$$

which is clearly linear.

④ Let U be an open neighborhood of a point $x_0 \in M$. There exists a smooth function $\rho : M \to [0,1]$ with compact support $\operatorname{supp}(\rho) := \overline{\{x \in M \mid \rho(x) \neq 0\}} \subset U$ which is identically 1 in an open neighborhood of x_0. We call ρ a <u>bump function</u> for (U, x_0).

⑤ Take $D \in \operatorname{der} C^\infty(V)$ and $x_0 \in M$. We define

$$D|_{x_0} : C^\infty(V) \to \mathbb{R}, \ f \mapsto (D(f))(x_0).$$

<u>Claim</u>: If $f_1, f_2 \in C^\infty(V)$ satisfy $f_1|_U = f_2|_U$ for an open set $U \subseteq V$ with $x_0 \in U$, then

$$D|_{x_0}(f_1) = D|_{x_0}(f_2).$$

<u>Proof</u>: Take a bump function ρ for (U, x_0).
Since $\rho \cdot (f_1 - f_2) \equiv 0$, we get that

$$
\begin{aligned}
0 &= D|_{x_0}\big(\rho \cdot (f_1 - f_2)\big) \\
&= D|_{x_0}(\rho)\,\underbrace{(f_1 - f_2)(x_0)}_{=0} + \underbrace{\rho(x_0)}_{=1}\,\underbrace{D|_{x_0}(f_1 - f_2)}_{= D|_{x_0}(f_1) - D|_{x_0}(f_2)}
\end{aligned}
$$

and hence the claim. $\square$

We then get a well-defined map:

- $(\Psi(D))(x_0) : C^\infty_{x_0}(\mathcal{M}) \to \mathbb{R}, \quad [f]_{x_0} \mapsto D|_{x_0}(\rho \cdot f|_V)$

  where, for a (U, f) representing $[f]_{x_0}$, ρ is any bump function for (U, x_0) and $\rho \cdot f \in C^\infty(V)$ is defined (in fact, well-defined) by

$$
(\rho \cdot f)(x) = \begin{cases} 0, & x \notin \operatorname{supp}(\rho) \\ \rho(x) f(x), & x \in U \end{cases} \qquad \forall x \in \mathcal{M}.
$$

- Clearly, $(\Psi(D))(x_0) \in T_{x_0}\mathcal{M}$.

6. Take $D \in \operatorname{der} C^\infty(V)$. Then the induced map

$$
\Psi(D) : V \to T\mathcal{M}, \quad x \mapsto (\Psi(D))(x)
$$

belongs to $\mathfrak{X}(V)$.

7. We get by 6 a linear map $\Psi : \operatorname{der} C^\infty(V) \to \mathfrak{X}(V)$, which satisfies $\Phi \circ \Psi = \mathrm{id}_{\operatorname{der} C^\infty(V)}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{X}(V)}$.

$\square$

2.10. Remark:

Like vector spaces underly linear algebra, vector bundles underly what can be seen as "parametrized" linear algebra. Indeed, various constructions for vector spaces can be generalized to that setting: let X be a Hausdorff topological space and let E and F be vector bundles over X (both real or complex) of dimension n and m, respectively.

(i) The Whitney sum (or direct sum bundle) $E \oplus F$ is the vector bundle of dimension $n+m$ with

$$(E \oplus F)_x = E_x \oplus F_x \quad \forall x \in X.$$

(ii) The tensor product bundle $E \otimes F$ is the vector bundle of dimension $n \cdot m$ with

$$(E \otimes F)_x = E_x \otimes F_x \quad \forall x \in X.$$

(iii) The homomorphisms bundle $\hom(E,F)$ is the vector bundle of dimension $n \cdot m$ with

$$\hom(E,F)_x = \hom(E_x, F_x) \quad \forall x \in X.$$

Analogously, the dual bundle E^* (see Exercise 2A-1(i)), the exterior product $\Lambda^p E$, the vector bundle $\mathrm{mult}^p(E)$ of all p-multilinear maps, and the vector bundles $\mathrm{sym}^p(E)$ and $\mathrm{alt}^p(E)$ of all symmetric respectively alternating p-multilinear maps can be defined.

2.11. Definition (Riemannian metric)

Let M be a smooth manifold of dimension n. A Riemannian metric on M is a smooth section g of the vector bundle $\mathrm{sym}^2(TM)$ such that

$$g_x : T_x M \times T_x M \to \mathbb{R}$$

is an inner product on $T_x M$ for each $x \in X$.

2.12 Remark:

(i) A topological space X is said to be paracompact if every open cover $(U_i)_{i\in I}$ of X has an open refinement $(V_j)_{j\in J}$ (i.e., $(V_j)_{j\in J}$ is an open cover of X and $\forall j \in J \ \exists i \in I : \ V_j \subseteq U_i$) that is locally finite (i.e., every $x \in X$ has a neighborhood V such that

$$\#\{j \in J \mid V \cap V_j \neq \emptyset\} < \infty).$$

(ii) Let X be a topological space. A partition of unity on X is a family $(\rho_i)_{i\in I}$ of continuous functions $\rho_i : X \to [0,1]$ such that

- each $x \in X$ has an open neighborhood V such that
$$\#\{i \in I \mid \rho_i|_V \neq 0\} < \infty;$$
- $\forall x \in X : \ \sum_{i\in I} \rho_i(x) = 1.$

We say that $(\rho_i)_{i\in I}$ is <u>subordinate</u> to an open cover $(U_i)_{i\in I}$ of X, if $\operatorname{supp}(\rho_i) \subseteq U_i$ for all $i \in I$.

(iii) On a paracompact space X, each open cover $(U_i)_{i\in I}$ of X has a subordinate partition of unity $(\rho_i)_{i\in I}$.
If $X = M$ is a paracompact smooth manifold, then each ρ_i can be chosen to be smooth.

<u>2.13. Theorem:</u>

Let M be a smooth manifold of dimension n. Suppose that M is paracompact. Then there exists a Riemannian metric on M.

<u>Proof:</u>

Take a bundle atlas $\mathcal{A} = \{(U_i, \tau_i) \mid i \in I\}$ of TM and let $(\rho_i)_{i\in I}$ be a smooth partition of unity subordinate to $(U_i)_{i\in I}$. We obtain a Riemannian metric by

$$g_x(\delta_1, \delta_2) := \sum_{i\in I} \rho_i(x) \langle \Theta_{i,x}^{-1}(\delta_1), \Theta_{i,x}^{-1}(\delta_2) \rangle$$

for all $x \in M$ and $\delta_1, \delta_2 \in T_x M$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. $\square$

2.14 Definition (differential forms):

Let M be a smooth manifold of dimension n.

(i) The dual bundle T^*M to the tangent bundle TM is called the cotangent bundle.

(ii) A smooth differential form of degree p (or p-form) is a smooth section of $\Lambda^p T^*M$. We put

$$\Omega^p(M) := \Gamma^\infty(M, \Lambda^p T^*M).$$

We call $\Omega^\bullet(M) := \bigoplus_{p\geq 0} \Omega^p(M)$ the exterior algebra.

(iii) The exterior derivative is the unique family $(d^p)_{p\geq 0}$ of $\mathbb{R}$-linear maps $d^p : \Omega^p(M) \to \Omega^{p+1}(M)$ satisfying

- $\forall f \in \Omega^0(M) = C^\infty(M)$ $\forall x \in M$
  $(d^0 f)(x) : T_x M \to \mathbb{R}, \quad \delta \mapsto \delta([f]_x)$;
- $\forall p \geq 0 : d^{p+1} \circ d^p = 0$;
- $\forall \omega \in \Omega^p(M), \eta \in \Omega^q(M)$:
  $$d^{p+q}(\omega \wedge \eta) = d^p(\omega) \wedge \eta + (-1)^p \cdot \omega \wedge d^q(\eta),$$
  where $\wedge$ on $\Omega^\bullet(M)$ is defined pointwise, i.e., $(\omega \wedge \eta)(x) := \omega(x) \wedge \eta(x)$ in $\Lambda^{p+q} T_x^* M$ for each $x \in M$.

(iv) Let $T^*M_{\mathbb{C}}$ be the complexification of T^*M, i.e. $T^*M_{\mathbb{C}} := TM \otimes (M \times \mathbb{C})$, where $M \times \mathbb{C}$ is the real trivial bundle (i.e., $\pi : M \times \mathbb{C} \to M, (x, \lambda) \mapsto x$) of dimension 2 ($\mathbb{C} = \mathbb{R}^2$).

We put $\Omega^p_{\mathbb{C}}(M) := \Gamma^\infty(M, \Lambda^p_{\mathbb{C}} T^* M_{\mathbb{C}})$.