Note that \( \text{der} \, C^\infty(V) \) denotes the space of derivations on \( C^\infty(V) \), i.e., linear maps \( D : C^\infty(V) \rightarrow C^\infty(V) \) that satisfy the product rule
\[
D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \quad \forall f, g \in C^\infty(V).
\]

**Proof:**

1. Take \( X \in \mathfrak{X}(V) \) and \( f \in C^\infty(V) \).

   **Claim:** \( \rho : V \rightarrow \mathbb{R}, \, x \mapsto X(x)(\langle f \rangle_x) \) is smooth.

   **Proof:** Consider the bundle atlas \( \mathcal{U} = \{(U_i, \tau_i) \mid i \in I\} \) that we introduced in Definition 2.6. It suffices to show that \( \rho |_{U_i \cap V} \) is smooth for all \( i \in I \).

   Fix \( i \in I \). By Definition 2.7 (ii), we find a smooth function \( \tilde{g} = (g_1, \ldots, g_n) : U_i \cap V \rightarrow \mathbb{R}^n \) such that \( \tau_i(\tilde{X}(x)) = (x, \tilde{g}(x)) \) for all \( x \in U_i \cap V \).

   Thus, \( \tilde{X}(x) = \tau_i^{-1}(x, \tilde{g}(x)) = (x, \Theta_i(x, \tilde{g}(x))) \) and
   \[
   \rho(x) = \tilde{X}(x)(\langle f \rangle_x) = \Theta_i(x, \tilde{g}(x)) (\langle f \rangle_x)
   = \sum_{\delta=1}^{n} g_\delta(x) (\partial_\delta (f \circ \tau_i^{-1}))(\tau_i(x)) \quad \forall x \in U_i \cap V,
   \]
   which shows that \( \rho \) is smooth on \( U_i \cap V \).

2. Take \( X \in \mathfrak{X}(V) \). By (1), we have a well-defined map.
Claim: \( \Phi(X) \) is a derivation.

Proof: Let \( f, g \in C^\infty(V) \) be given. Then, for all \( x \in U \),
\[
(\Phi(X)(f \cdot g))(x) = X(x)([f]_x \cdot [g]_x)
= X(x)([f]_x) \cdot g(x) + f(x) \cdot X(x)([g]_x)
= (\Phi(X)(f) \cdot g + f \cdot \Phi(X)(g))(x).
\]

(3) By (2), we have a well-defined map
\[
\Phi: \mathfrak{X}(V) \to \text{Der} C^\infty(V), \quad X \mapsto \Phi(X),
\]
which is clearly linear.

(4) Let \( U \) be an open neighborhood of a point \( x_0 \in M \). There exists a smooth function \( \rho: M \to [0,1] \) with compact support \( \text{supp} \rho := \{ x \in U | \rho(x) \neq 0 \} \subset U \) which is identically 1 in an open neighborhood of \( x_0 \). We call \( \rho \) a bump function for \( (U, x_0) \).

(5) Take \( D \in \text{Der} C^\infty(V) \) and \( x_0 \in M \). We define
\[
D|_{x_0} : C^\infty(V) \to \mathbb{R}, \quad f \mapsto (D(f))(x_0).
\]
Claim: If \( f_1, f_2 \in C^\infty(V) \) satisfy \( f_1 u = f_2 u \)
for an open set \( U \subset V \) with \( x_0 \in U \), then
\[
D|_{x_0}(f_1) = D|_{x_0}(f_2).
\]
Proof: Take a bump function \( \varphi \) for \((U, x_0)\). Since \( \varphi \cdot (f_1 - f_2) \equiv 0 \), we get that
\[
0 = D|_{x_0}(\varphi \cdot (f_1 - f_2)) = D|_{x_0}(\varphi) (f_1 - f_2)(x_0) + \varphi(x_0)D|_{x_0}(f_1 - f_2) = \underline{0} = D|_{x_0}(f_1) - D|_{x_0}(f_2)
\]
and hence the claim.

We thus get a well-defined map
\[
(\Psi(D))_{x_0} : C^\infty_{x_0}(U) \to \mathbb{R}, \quad [f]_{x_0} \mapsto D|_{x_0}(\varphi \cdot f)_V
\]
where, for \((U, f)\) representing \([f]_{x_0}\), \(\varphi\) is any bump function for \((U, x_0)\) and \(\varphi \cdot f \in C^\infty(U)\) is defined (in fact well-defined) by
\[
(\varphi \cdot f)(x) = \begin{cases} 0 & x \not\in \text{supp}\,(\varphi), \\ \varphi(x)f(x) & x \in U \end{cases} \quad \forall x \in U.
\]

- Clearly, \((\Psi(D))_{x_0} \in T_{x_0}U\).

(6) Take \(D \in \text{der}\,C^\infty(U)\). Then the induced map
\[
\Psi(D) : V \to TUM, \quad x \mapsto (\Psi(D))(x) \text{ belongs to } \mathcal{F}(V).
\]

(7) We get by (6) a linear map \(\Psi : \text{der}\,C^\infty(U) \to \mathcal{F}(V)\), which satisfies \(\overline{D} \circ \Psi = \text{id}_{\text{der}\,C^\infty(U)}\) and \(\Psi \circ \overline{D} = \text{id}_{\mathcal{F}(V)}\).
Like vector spaces, underlie linear algebra, vector bundles, whereby what can be seen as "parametrized" linear algebra. Indeed, various constructions for vector spaces can be generalized to that setting; let $X$ be a Hausdorff topological space and let $E$ and $F$ be vector bundles over $X$ (both real or complex) of dimension $n$ and $m$, respectively.

(i) The Whitney sum (or direct sum bundle) $E \oplus F$

in the vector bundle of dimension $n+m$ with

$$(E \oplus F)_x = E_x \oplus F_x \quad \forall x \in X.$$ 

(ii) The tensor product bundle $E \otimes F$ is the vector bundle of dimension $n \cdot m$ with

$$(E \otimes F)_x = E_x \otimes F_x \quad \forall x \in X.$$ 

(iii) The homomorphism bundle $\text{hom}(E, F)$ is the vector bundle of dimension $n \cdot m$ with

$$\text{hom}(E, F)_x = \text{hom}(E_x, F_x) \quad \forall x \in X.$$ 

Analogously, the dual bundle $E^*$ (see Exercise 2A-1(i)), the exterior product $\Lambda^p E$, the vector bundle $\text{mult}^p(E)$ of all $p$-multilinear maps, and the vector bundles $\text{sym}^p(E)$ and $\text{alt}^p(E)$ of all symmetric respectively alternating $p$-multilinear maps can be defined.
2.11 Definition (Riemannian metric)

Let \( M \) be a smooth manifold of dimension \( n \).

A **Riemannian metric** on \( M \) is a smooth section \( g \) of the vector bundle \( 
\text{Sym}^2 (TM) \) such that

\[
g_x : T_x M \times T_x M \to \mathbb{R}
\]

is an inner product on \( T_x M \) for each \( x \in X \).

2.12 Remark:

(i) A topological space \( X \) is said to be **paracompact** if every open cover \( (U_i)_{i \in I} \) of \( X \) has an **open refinement** \( (V_j)_{j \in J} \) (i.e., \( (V_j)_{j \in J} \) is an open cover of \( X \) and \( \forall j \in J \exists i \in I : V_j \subseteq U_i \)) that is **locally finite** (i.e., every \( x \in X \) has a neighborhood \( V \) such that

\[
\# \{ j \in J \mid V \cap V_j \neq \emptyset \} < \infty
\]

(ii) Let \( X \) be a topological space. A **partition of unity** on \( X \) is a family \( (\varphi_i)_{i \in I} \) of continuous functions \( \varphi_i : X \to [0,1] \) such that

- each \( x \in X \) has an open neighborhood \( V \) such that

\[
\# \{ i \in I \mid \varphi_i |_V \neq 0 \} < \infty;
\]

- \( \forall x \in X : \sum_{i \in I} \varphi_i (x) = 1. \)
We say that \((S_i)_{i \in I}\) is \underline{subordinate} to an open cover \((U_i)_{i \in I}\) of \(X\), if \(\text{supp}(S_i) \subseteq U_i\) for all \(i \in I\).

(iii) On a paracompact space \(X\), each open cover \((U_i)_{i \in I}\) of \(X\) has a subordinate partition of unity \((S_i)_{i \in I}\). If \(X = U\) is a paracompact smooth manifold, then each \(S_i\) can be chosen to be smooth.

2.13. Theorem:

Let \(U\) be a smooth manifold of dimension \(n\). Suppose that \(U\) is paracompact. Then there exists a Riemannian metric on \(U\).

\textbf{Proof:}

Take a bundle atlas \(A = \{(U_i, \tau_i) | i \in I\}\) of \(TU\) and let \((S_i)_{i \in I}\) be a smooth partition of unity subordinate to \((U_i)_{i \in I}\). We obtain a Riemannian metric by

\[
g_x(S_1, S_2) := \sum_{i \in I} c_i(x) \left< \Theta_i, x(S_1), \Theta_i, x(S_2) \right>
\]

for all \(x \in U\) and \(S_1, S_2 \in T_x U\), where \(<\cdot, \cdot>\) is the standard inner product on \(\mathbb{R}^n\). \(\square\)
2.14 Definition (differential forms):

Let $\mathcal{M}$ be a smooth manifold of dimension $n$.

(i) The dual bundle $T^*\mathcal{M}$ to the tangent bundle $T\mathcal{M}$ is called the cotangent bundle.

(ii) A smooth differential form of degree $p$ (or $p$-form) is a smooth section of $\Lambda^p T^*\mathcal{M}$. We put

$$\Omega^p(\mathcal{M}) := \Gamma^\infty(\mathcal{M}, \Lambda^p T^*\mathcal{M}).$$

We call $\Omega^*(\mathcal{M}) := \bigoplus_{p \geq 0} \Omega^p(\mathcal{M})$ the exterior algebra.

(iii) The exterior derivative $d$ is the unique family $(d^p)_{p \geq 0}$ of $\mathbb{R}$-linear maps $d^p : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ satisfying

- $\forall f \in \Omega^0(\mathcal{M}) = C^\infty(\mathcal{M}) \quad \forall x \in \mathcal{M}$
  $$(d^0 f)(x) : T_x \mathcal{M} \to \mathbb{R}, \quad s \mapsto s([f]_x);$$
- $\forall p \geq 0 : \quad d^{p+1} \circ d^p = 0$;
- $\forall \omega \in \Omega^p(\mathcal{M}), \gamma \in \Omega^q(\mathcal{M})$:
  $$d^{p+q}(\omega \wedge \gamma) = d^p(\omega) \wedge \gamma + (-1)^p \cdot \omega \wedge d^q(\gamma),$$
  where $\wedge$ on $\Omega^*(\mathcal{M})$ is defined pointwise, i.e.,
  $$(\omega \wedge \gamma)(x) := \omega(x) \wedge \gamma(x) \text{ in } \Lambda^{p+q} T^*_x \mathcal{M} \text{ for each } x \in \mathcal{M}.$$  

(iv) Let $T^*_\mathbb{C}\mathcal{M}$ be the complexification of $T^*\mathcal{M}$, i.e.

$$T^*_\mathbb{C}\mathcal{M} := T\mathcal{M} \otimes (\mathcal{M} \times \mathbb{C}),$$

where $\mathcal{M} \times \mathbb{C}$ is the real trivial bundle (i.e., $\pi : \mathcal{M} \times \mathbb{C} \to \mathcal{M}, (x, \lambda) \mapsto x$) of dimension $2$ ($\mathbb{C} = \mathbb{R}^2$).

We put $\Omega^p(\mathbb{C}\mathcal{M}) := \Gamma^\infty(\mathcal{M}, \Lambda^p T^*_\mathbb{C}\mathcal{M}).$