INTRODUCTION TO NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

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1. Introduction

Several "classical theories" in mathematics can be extended to the noncommutative world. The appropriate framework is often obtained by the following recipe:

(i) take a "classical space", i.e., a set X endowed with some additional structure (e.g., topological or measure spaces, groups, manifolds, Lie groups, ...);

(ii) consider a suitable algebra of functions over that space (e.g., $C_0(X)$, $C(X)$, $L^\infty(X)$, $C^\infty(X)$, ...);

(iii) transfer the additional structure of the space to its associated commutative algebra of functions and provide an intrinsic characterization of that structure;

(iv) drop the assumption of commutativity.

Finding a good axiomatic description in (iii) that allows one to perform step (iv) is clearly the core problem and by no means straightforward.
The right choice confirms itself by a "reconstruction theorem" by which, in the commutative case, the

underlying space can be "recovered" from that set of axioms. We list some prominent examples:

classical space	noncommutative counterpart	reconstruction theorem
locally compact (compact) Hausdorff topological space	(unital) C^*-algebra n.c. topology	Gelfand-Naimark (FA I, Cor. 10.17)
compact Hausdorff topological space with finite Radon measure	von Neumann algebra n.c. measure theory	FA II, Thm 8.15
compact topological group	compact quantum group "n.c." group theory	Tannaka - Krein
compact oriented smooth manifold	spectral triple / (unbounded) K-cycle (Connes, 1994) n.c. differential geometry	Connes' reconstruction theorem (2013)

The actual "noncommutative space" is mostly just a "virtual" object behind those algebras. The classical theories are thus rebuilt in an algebraic way, imitating the dual picture on their associated algebras of functions.

This philosophy underlies also the theory of noncommutative differential geometry that Alain Connes began to develop around the 80's. His motivation was to extend classical tools to

- spaces that are badly behaved as point sets but correspond naturally to (noncommutative) algebras (e.g., Penrose tilings, the space of leaves of a foliation, the phase space in quantum mechanics, ...)
- general noncommutative situations without an underlying space.

But even for classical situations that are purely commutative, this point of view gives new insights. Within noncommutative differential geometry, manifolds are studied by some spectral data. The following definition is at the heart of that approach.

1.1. Definition (Connes, 1994)

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where

- $\mathcal{A}$ is a unital complex $*$-algebra,
- $\mathcal{H}$ is a separable complex Hilbert space with a faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$,
- D is a (possibly unbounded) selfadjoint operator on $\mathcal{H}$, say $D : \mathcal{H} \supseteq \operatorname{dom} D \to \mathcal{H}$, with compact resolvent, i.e.,

$$(D - \lambda 1)^{-1} \in K(\mathcal{H}) \qquad \forall \lambda \in \mathbb{C} \setminus \sigma(D),$$

such that the following holds for all $a \in \mathcal{A}$:

We have that $\pi(a) \operatorname{dom} D \subseteq \operatorname{dom} D$ and the commutator $[D, \pi(a)] = D\pi(a) - \pi(a)D$ extends to an operator in $B(\mathcal{H})$.

1.2. Example:

Consider on $\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$ the arc *length measure* m, i.e., the push-forward of the Lebesgue measure on $\mathbb{R}$ via the map $\gamma : \mathbb{R} \to \mathbb{T}$, $t \mapsto e^{it}$.

A function $f : \mathbb{T} \to \mathbb{C}$ is differentiable if and only if $f \circ \gamma : \mathbb{R} \to \mathbb{C}$ is so; its derivative $f' : \mathbb{T} \to \mathbb{C}$

is determined by

$$f'(\gamma(t))\,\gamma'(t) = (f\circ\gamma)'(t) \quad \forall t\in\mathbb{R}.$$

Take $\mathcal{H} = L^2(\mathbb{T}, m)$ and $\mathcal{A} = C^\infty(\mathbb{T})$ with the $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by

$$\pi(f) := M_f : L^2(\mathbb{T}, m) \to L^2(\mathbb{T}, m), \quad g \mapsto fg.$$

Further, we consider the densely defined operator

$$D_0 : \mathcal{H} \supseteq \operatorname{dom} D_0 \to \mathcal{H}, \quad g \mapsto \frac{1}{i} g'$$

on $\operatorname{dom} D_0 := C^1(\mathbb{T})$, which is a symmetric operator.
One can show that D_0 is essentially selfadjoint; let D be its closure, which is thus selfadjoint. Then $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Indeed, if $\mathcal{F} : \mathcal{H} \to \ell^2(\mathbb{Z})$ is the Fourier transform, i.e.

$$\mathcal{F} : L^2(\mathbb{T}, m) \to \ell^2(\mathbb{Z}), \quad f \mapsto (\hat{f}_n)_{n\in\mathbb{Z}},$$

$$\hat{f}_n := \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\,\zeta^{-n}\,dm(\zeta), \quad n\in\mathbb{Z},$$

then $\mathcal{F} D \mathcal{F}^{-1}$ is the multiplication by $(n)_{n\in\mathbb{Z}}$, hence $(D-\lambda)^{-1} \in K(\mathcal{H})$ for all $\lambda \in \mathbb{C}\setminus\mathbb{Z}$; moreover,

$$[D, \pi(f)]\,g = D_0(fg) - f D_0 g = \frac{1}{i}\pi(f')\,g$$

for all $f\in\mathcal{A}$, $g \in C^1(\mathbb{T})$.

We will see that more general manifolds $\mathcal{M}$ induce spectral triples in a similar way. Much of the structure of $\mathcal{M}$ can be recovered from $(\mathcal{A}, \mathcal{H}, \mathcal{D})$:

- $d(p, q) := \sup\{ |f(p) - f(q)| \mid f \in \mathcal{A} : \|[\mathcal{D}, \pi(f)]\| \leq 1 \}$

  is the geodesic distance between $p, q \in \mathcal{M}$.

- $\int_{\mathcal{M}} f \, d\mathrm{vol} = c(n) \, \mathrm{Tr}(f |\mathcal{D}|^{-n}) \quad \forall f \in \mathcal{A}$