4.16 Example:

In the situation of Example 4.13, we see that

$$P := f \cdot \tilde{\Delta}^{-1/2} \in \Psi^{-1}(\mathbb{T}, T^*\mathbb{T}_{\mathbb{C}}) \text{ with } \sigma^P(x,\xi) = f(x)\,\|\xi\|^{-1}$$

and thus

$$\mathrm{Res}_W(P) = \frac{1}{2\pi}\int_{S^*\mathbb{T}} \operatorname{tr}\sigma^P(x,\xi)\,\omega_\xi \wedge dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\underbrace{\left(\int_{\|\xi\|_{T_x^*\mathbb{T}}=1}\omega_\xi\right)}_{=\,2} dx = \frac{1}{\pi}\int_{\mathbb{T}} f(\zeta)\,dm(\zeta).$$

Therefore, Theorem 4.15 yields (4.5).

4.17. Remark:

(i) Let $(\mathcal{A}, \mathcal{H}, D)$ be the Hodge-de Rham triple (Thm 2.19) for an oriented compact Riemannian manifold (M, g) of dimension n. Then the Hodge-Laplacian $\Delta = D^2$ is a pseudodifferential operator of order 2. After modification on ker Δ, we may suppose that Δ is invertible; then, for each $f \in C^\infty(M, \mathbb{C}) = \mathcal{A}$, $P := f \cdot \Delta^{-n/2} \in \Psi^{-n}(M, \Lambda^\bullet_{\mathbb{C}} T^*M_{\mathbb{C}})$ and by Theorem 4.15,

$$⨍ f \cdot \Delta^{-n/2} = \frac{1}{n}\mathrm{Res}_W(f \cdot \Delta^{-n/2}) = c(n)\int_M f,$$

for some constant $c(n) > 0$.

(ii) A similar result is true for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$

associated to spin^c manifolds, where D is the Dirac operator; see Remark 2.21.

(iii) In view of results like (i) and (ii), one considers $\Delta^{-n/2}$ for $\Delta := D^2$ as the counterpart of the Riemann-Lebesgue measure for general spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where $n > 0$ is chosen such that $\Delta^{-n/2} \in \mathcal{I}_1(\mathcal{H})$; if such n exists, we call it the <u>dimension</u> of $(\mathcal{A}, \mathcal{H}, D)$.

$\frown$

<u>4.18. Example</u> (Laplace - Beltrami operator)

In Theorem 2.17., the Riemann Lebesgue measure vol was constructed for every compact Riemannian manifold (M, g), namely as the unique Radon measure vol on $\mathcal{B}(M)$, the Borel σ-algebra of M, which satisfies

$\frown$

$$\int_M f = \int_M f(x)\, d\mathrm{vol}(x) \qquad \forall\ f \in C(M, \mathbb{C})$$

This relies on the Riesz-Markov-Kakutani representation theorem, which one applies to the positive linear functional

$$\int_M : C(M, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_M f$$

that is given by (linear and continuous) extension of

$$\int_M : C^\infty(M) \to \mathbb{R}, \quad f \mapsto \int_M f .$$

The Hodge-de Rham triple (Theorem 2.19), however, which recovers integration as we have seen in Remark 4.17, requires in addition an orientation on $\mathcal{M}$. In order to deal with the general case, one considers instead the <u>Laplace-Beltrami operator</u> $\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, which is determined by the condition

$$\int_{\mathcal{M}} f(x)(\Delta g)(x)\, d\mathrm{vol}(x) = -\langle df, dg\rangle_{\Omega^1(\mathcal{M})}$$

for every $f, g \in C^\infty(\mathcal{M})$; Δ is a differential operator of order 2 with principal symbol $\sigma^\Delta(x,\xi) = -\|\xi\|^2_{T_x^*\mathcal{M}_{\mathbb{C}}}$ and hence elliptic. Therefore, after modification of Δ on its kernel, we may consider $\Delta^{-n/2}$, which is then a pseudodifferential operator of order $-n$.

Theorem 4.15 yields that $\Delta^{-n/2} \in \mathcal{L}^{(1,\infty)}(L^2(\mathcal{M}))$, which is in accordance with <u>Weyl's law</u> (1911): if

$$0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le \lambda_{n+1}, \quad \lambda_n \to \infty$$

are the eigenvalues of Δ (listed with multiplicities) and

$$N(\lambda) := \#\{ j \in \mathbb{N} \mid \lambda_j \le \lambda \} \quad \text{for } \lambda \ge 0,$$

then $\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\Omega_n}{n(2\pi)^n}\mathrm{vol}(\mathcal{M})$ with $\Omega_n := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$, where Γ is the <u>Gamma function</u> and Ω_n the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. Furthermore,

$$\mu_k(\Delta^{-n/2}) \sim \frac{\Omega_n}{n(2\pi)^n}\mathrm{vol}(\mathcal{M})\frac{1}{k} \quad \text{as } k \to \infty$$

for the characteristic values of $\Delta^{-n/2}$, i.e., $\Delta^{-n/2} \in \mathcal{I}_1(L^2(M))$.

Using Theorem 4.15, we obtain for every $f \in C^\infty(M, \mathbb{C})$

$$⨍ f \, \Delta^{-n/2} = \frac{\Omega_n}{n(2\pi)^n} \int_M f(x) \, d\,vol(x).$$

Indeed, $P := f \cdot \Delta^{-n/2} \in \Psi^{-n}(M; \mathbb{C})$ and

$$\sigma^P(x, \xi) = f(x) \, \|\xi\|^{-n}_{T_x^* M_{\mathbb{C}}},$$

so that

$$\begin{aligned} \operatorname{Res}_W(P) &= \frac{1}{(2\pi)^n} \int_{S^* M} \operatorname{tr} \sigma^P(x, \xi) \, \omega_\xi \wedge dx \\ &= \frac{1}{(2\pi)^n} \int_M \underbrace{\Big(\int_{\|\xi\|_{T_x^* M} = 1} \omega_\xi \Big)}_{= \Omega_n} f(x) \, dx \\ &= \frac{\Omega_n}{(2\pi)^n} \int_M f(x) \, d\,vol(x). \end{aligned}$$

<u>4.19 Remark:</u>

(i) We have used implicitely that M carries an orientation when we defined the Wodzicki residue via the forms ω_ξ and dx; in general, one replaces them by their <u>densities</u> $|\omega_\xi|$ and $|dx|$, which give the surface measure on $\{\xi \in T_x^* M \mid \|\xi\|_{T_x^* M} = 1\}$ and the Riemann-Lebesgue measure vol, respectively. Hence,

$$\operatorname{Res}_W(P) = \frac{1}{(2\pi)^n} \int_M \Big(\int_{\|\xi\|_{T_x^* M} = 1} \operatorname{tr} \sigma^P(x, \xi) \, |\omega_\xi| \Big) \, |dx|.$$

(ii) Furthermore, we made use of the general fact that an elliptic (pseudo-) differential operator $P \in \Psi^m(M;E)$ has an inverse modulo <u>smoothing operators</u> $\Psi^{-\infty}(M;E)$, which are those pseudodifferential operators that have symbols in the class

$$\mathrm{Sym}^{-\infty}(M;E) := \bigcap_{m \in \mathbb{R}} \mathrm{Sym}^m(M;E);$$

more explicitly, there is $Q \in \Psi^{-m}(M;E)$ such that

$$PQ - 1,\ QP - 1 \in \Psi^{-\infty}(M;E);$$

we call such Q a <u>parametrix</u> for P.

Note that $P \in \Psi^m(M;E)$ is said to be <u>elliptic</u> if its principal symbol σ^P has a representative, which is pointwise invertible whenever $\xi \in T_x^*M \setminus \{0\}$.

5. Tame spectral triples and integration

In Chapter 4, we have seen that, for spectral triples $(\mathcal{A}, \mathcal{H}, D)$ associated to compact Riemannian manifolds (M, g) of dimension n, $⨍ f \cdot \Delta^{-n/2}$ with $\Delta = D^2$ yields an operator-theoretic avatar of integration $\int_M f \, d\mathrm{vol}$ for functions $f \in C^\infty(M, \mathbb{C})$.

For a general spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of dimension $n > 0$ (or, more generally, (n, ∞)-summable), we may consider

$$\mathrm{Tr}_\omega(\pi(a)\Delta^{-n/2}) \qquad \text{for } a \in \mathcal{A},$$

for any Dixmier trace Tr_ω; see Theorem 4.8. We want to show that such "integrals" for elements in $\mathcal{A}$ inherit some properties of their commutative ancestors.

5.1. Theorem:

Let $(\mathcal{A}, \mathcal{H}, D)$ be an (n, ∞)-summable spectral triple (with a separable complex Hilbert space $\mathcal{H}$ of infinite dimension). Further, let $\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C}$ be any Dixmier trace. Then, for all $T \in B(\mathcal{H})$,

$$\mathrm{Tr}_\omega(\pi(a) T \Delta^{-n/2}) = \mathrm{Tr}_\omega(T \pi(a) \Delta^{-n/2}).$$

In particular,

$$\tau : \mathcal{A} \to \mathbb{C}, \quad a \mapsto \mathrm{Tr}_\omega(\pi(a)\Delta^{-n/2})$$

is a trace on $\mathcal{A}$, i.e., $\tau(a_1 a_2) = \tau(a_2 a_1)$ for all $a_1, a_2 \in \mathcal{A}$.