5.12 Definition (Connes, Moscovici, 1995)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a regular and n-summable spectral triple. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has discrete dimension spectrum if there is a discrete set $\mathcal{F} \subseteq \mathbb{C}$ with the following property: for every $P \in \Psi^0(\mathcal{A})$, the zeta function $\zeta_{P,\Delta} : \{z \in \mathbb{C} \mid \mathrm{Re}(z) > n\} \to \mathbb{C}$, $z \mapsto \mathrm{Tr}(P \Delta^{-z/2})$ extends to a meromorphic function on $\mathbb{C}$ with all poles contained in $\mathcal{F}$.

5.13. Example

Consider the operator $\tilde{\Delta}$ from Example 4.13. Then

$$\zeta_{1,\tilde{\Delta}}(z) = \mathrm{Tr}(\tilde{\Delta}^{-z/2}) = \sum_{n=0}^{\infty} \mu_n(\tilde{\Delta}^{-1/2})^z$$

$$= 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + 2\zeta(z),$$

where $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is the Dirichlet series for the Riemann zeta function; the latter admits a meromorphic extension to $\mathbb{C}$ with a simple pole at 1. Note that $\mathrm{Res}(\zeta, 1) = 1$ and hence

$$\mathrm{Res}(\zeta_{1,\tilde{\Delta}}, 1) = 2 = ⨍ \tilde{\Delta}^{-1/2}.$$

In fact, whenever $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$ is given and $\lim_{s \searrow 1} (s-1)\mathrm{Tr}(T^s)$ exists, then $T \in \mathcal{M}(\mathcal{H})$ and

$$⨍ T = \lim_{s \searrow 1^+} (s-1)\mathrm{Tr}(T^s).$$

6. Noncommutative differential forms

In Definition 2.14, we introduced the exterior algebra

$$\Omega^\bullet(\mathcal{M}) = \bigoplus_{p\geq 0} \Omega^p(\mathcal{M}), \quad \Omega^p(\mathcal{M}) = \Gamma^\infty(\mathcal{M}, \Lambda^p T^*\mathcal{M})$$

and the exterior derivative

$$d = (d^p)_{p\geq 0}, \quad d^p : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$$

for a smooth manifold $\mathcal{M}$. The goal of this chapter is to construct an analogue of $(\Omega^\bullet(\mathcal{M}), d)$ for spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

In the following, let $\mathcal{A}$ be a unital complex algebra.

6.1. Definition (universal differential forms):

(i) We put $\Omega^0(\mathcal{A}) := \mathcal{A}$.

(ii) Consider the multiplication map

$$\mu : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A}, \quad a_1 \otimes a_2 \mapsto a_1 a_2$$

and put $\Omega^1(\mathcal{A}) := \ker \mu \subseteq \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$.

(iii) For each $p \geq 2$, we define

$$\Omega^p(\mathcal{A}) := \Omega^1(\mathcal{A})^{\otimes_{\mathcal{A}} p} = \underbrace{\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})}_{p\text{-times}}.$$

Finally, we define the algebra of universal differential forms by

$$\Omega^\bullet(\mathcal{A}) := \bigoplus_{p\geq 0} \Omega^p(\mathcal{A}),$$

with the canonical algebra structure with multiplication given by $\otimes_{\mathcal{A}}$, i.e.,

$$\otimes_{\mathcal{A}} : \Omega^p(\mathcal{A}) \times \Omega^q(\mathcal{A}) \to \Omega^{p+q}(\mathcal{A}), (\omega, \eta) \mapsto \omega \otimes_{\mathcal{A}} \eta.$$

Elements of $\Omega^p(\mathcal{A})$ are called p-forms.

6.2 Remark:

(i) $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ becomes an $\mathcal{A}$-bimodule by

$$b_1 \cdot \omega \cdot b_2 := \sum_{i=1}^{N} (b_1 a_1^i) \otimes (a_2^i b_2)$$

for $\omega = \sum_{i=1}^{N} a_1^i \otimes a_2^i \in \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ and $b_1, b_2 \in \mathcal{A}$.

Since μ is an $\mathcal{A}$-bimodule map, i.e.,

$$\mu(b_1 \cdot \omega \cdot b_2) = b_1 \mu(\omega) b_2$$

for all $\omega \in \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ and $b_1, b_2 \in \mathcal{A}$, we see that also $\Omega^1(\mathcal{A})$ is an $\mathcal{A}$-bimodule (with the restricted action). Therefore, each $\Omega^p(\mathcal{A})$ for $p \geq 0$ forms an $\mathcal{A}$-bimodule.

(ii) There is a canonical isomorphism

$$(\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A})^{\otimes_{\mathcal{A}} p} \longrightarrow \mathcal{A}^{\otimes_{\mathbb{C}} (p+1)},$$

$$(a_1 \otimes b_1) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (a_n \otimes b_n) \mapsto a_1 \otimes (b_1 a_2) \otimes \cdots \otimes (b_{n-1} a_n) \otimes b_n.$$

Thus, we may view $\Omega^p(\mathcal{A}) \subseteq \mathcal{A}^{\otimes_{\mathbb{C}} (p+1)}$.

6.3. Definition (universal derivations)

We define a family $\delta = (\delta^p)_{p\geq 0}$ of linear maps by

$$\delta^0 : \mathcal{A} = \Omega^0(\mathcal{A}) \to \Omega^1(\mathcal{A}), \quad a \mapsto 1 \otimes a - a \otimes 1$$

and

$$\delta^p : \Omega^p(\mathcal{A}) \longrightarrow \Omega^{p+1}(\mathcal{A}),$$

$$\omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_p \longmapsto \sum_{k=1}^{p} (-1)^{k+1} \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \delta^0 \omega_k \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_p .$$

6.4 Remark:

(i) One can check that $\delta^{p+1} \circ \delta^p = 0$ for all $p \geq 0$ and

$$\delta^{p+q}(\omega \otimes_{\mathcal{A}} \eta) = \delta^p(\omega) \otimes_{\mathcal{A}} \eta + (-1)^p \omega \otimes_{\mathcal{A}} \delta^q(\eta)$$

for all $\omega \in \Omega^p(\mathcal{A})$ and $\eta \in \Omega^q(\mathcal{A})$.

(ii) Each $\omega \in \Omega^p(\mathcal{A})$ can be written in the form

$$\omega = \sum_{i=1}^{N} a_0^i \cdot \delta a_1^i \otimes_{\mathcal{A}} \delta a_2^i \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \delta a_p^i \qquad (6.1)$$

with $a_0^i, a_1^i, \dots, a_p^i \in \mathcal{A}$ for $i = 1, \dots, N$.

6.5. Definition (graded differential algebra)

A graded differential algebra $(\Gamma^\bullet, \partial)$ is a complex unital algebra $\Gamma^\bullet = \bigoplus_{p\geq 0} \Gamma^p$ whose product is graded in the sense that $\Gamma^p \Gamma^q \subseteq \Gamma^{p+q}$ for all $p, q \geq 0$, together with a differential ∂, i.e., a family $\partial = (\partial^p)_{p\geq 0}$ of linear maps $\partial^p : \Gamma^p \to \Gamma^{p+1}$ which satisfy $\partial^{p+1} \circ \partial^p = 0$ for all

$p \geq 0$ and

$$\partial^{p+q}(\omega\eta) = \partial^p(\omega)\eta + (-1)^p \omega\, \partial^q(\eta) \quad \forall \omega \in \Gamma^p, \eta \in \Gamma^q.$$

6.6. Theorem:

$(\Omega^\bullet(\mathcal{A}), \delta)$ is a graded differential algebra with the following universal property:

If $(\Gamma^\bullet, \partial)$ is a graded differential algebra and if $\rho : \Omega^0(\mathcal{A}) \to \Gamma^0$ is a morphism of unital algebras, then there exists a unique extension of ρ to a morphism $\tilde{\rho} = (\tilde{\rho}^p)_{p \geq 0} : \Omega^\bullet(\mathcal{A}) \to \Gamma^\bullet$ of graded algebras such that

$$\begin{tikzcd}
\Omega^p(\mathcal{A}) \arrow[r, "\tilde{\rho}^p"] \arrow[d, "\delta^p"'] & \Gamma^p \arrow[d, "\partial^p"] \\
\Omega^{p+1}(\mathcal{A}) \arrow[r, "\tilde{\rho}^{p+1}"] & \Gamma^{p+1}
\end{tikzcd} \qquad (6.2)$$

commutes for each $p \geq 0$.

Thus, we call $(\Omega^\bullet(\mathcal{A}), \delta)$ the <u>universal differential algebra of forms on $\mathcal{A}$</u>.

Proof:

① Uniqueness: Take any $\omega \in \Omega^p(\mathcal{A})$, written like in (6.1); then

$$\begin{aligned}
\tilde{\rho}^p(\omega) &= \sum_{i=1}^{N} \tilde{\rho}^0(a_0^i)\,\tilde{\rho}^1(\delta a_1^i) \cdots \tilde{\rho}^1(\delta a_p^i) \\
&= \sum_{i=1}^{N} \rho(a_0^i)\, \partial^1(\rho(a_1^i)) \cdots \partial^1(\rho(a_p^i)), \qquad (6.3)
\end{aligned}$$

which shows that $\tilde{\varrho}^p$ is uniquely determined by ϱ.

② Existence: Define $\tilde{\varrho}^p$ by (6.3) and verify that they give morphisms for which (6.2) commutes.

$\square$

6.7. Remark:

Suppose now that $\mathcal{A}$ is a $*$-algebra. Then $\Omega^\bullet(\mathcal{A})$ becomes a $*$-algebra with respect to the induced involution defined by

- $(\delta a)^* := -\delta(a^*) \qquad$ for all $a \in \mathcal{A}$,
- $(a_0 \cdot \delta a_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \delta a_p)^* := (\delta a_p)^* \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (\delta a_1)^* \cdot a_0^*$

$$= a_p^* \cdot (\delta a_{p-1}^*) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (\delta a_0^*)$$

$$+ \sum_{k=0}^{p-1} (-1)^{p+k} (\delta a_p^*) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \delta(a_{k+1}^* a_k^*) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (\delta a_0^*)$$

for $a_0, a_1, \dots, a_p \in \mathcal{A}$.

This corresponds to

- $(a_1 \otimes a_2)^* = a_2^* \otimes a_1^*$ on $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$
- $(\omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_p)^* = \omega_p^* \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_1^*$ on $(\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A})^{\otimes_{\mathcal{A}} p}$.

6.8. Theorem (universal property of $\delta^0 : \mathcal{A} \to \Omega^1(\mathcal{A})$):

Let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $\partial : \mathcal{A} \to \mathcal{M}$ be a derivation (i.e., a $\mathbb{C}$-linear map which satisfies the