Leibniz rule $\partial(a_1 a_2) = \partial(a_1)\cdot a_2 + a_1 \cdot \partial(a_2)$ for all $a_1, a_2 \in \mathcal{A}$), then there exists a unique morphism $\rho: \Omega^1(\mathcal{A}) \to \mathcal{M}$ of bimodules such that the diagram

$$\begin{tikzcd} \Omega^1(\mathcal{A}) \arrow[dr, dashed, "\rho"] & \\ \mathcal{A} \arrow[u, "\delta^0"] \arrow[r, "\partial"] & \mathcal{M} \end{tikzcd}$$

commutes.

Proof: Analogous to Theorem 6.6. □

From now on, consider a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.
We apply Theorem 6.8 to the $\mathcal{A}$-bimodule $\mathcal{B}(\mathcal{H})$ with

$$a_1 \cdot T \cdot a_2 := \pi(a_1) T \pi(a_2) \qquad \forall\, a_1, a_2 \in \mathcal{A},\ T \in \mathcal{B}(\mathcal{H})$$

and the derivation

$$\partial : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \quad a \mapsto [\mathcal{D}, \pi(a)].$$

This gives a unique morphism $\pi^1 : \Omega^1(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ of bimodules satisfying $\partial = \pi^1 \circ \delta^0$, i.e., for $a_0, a_1 \in \mathcal{A}$,

$$\pi^1(a_0\, \delta a_1) = \pi(a_0)\pi^1(\delta a_1) = \pi(a_0)\, \partial a_1 = \pi(a_0)[\mathcal{D}, \pi(a_1)].$$

We extend π^1 multiplicatively to a morphism

$$\pi_{\mathcal{D}} = (\pi^p)_{p \geq 0} : \Omega^\bullet(\mathcal{A}) \to \mathcal{B}(\mathcal{H}),$$

i.e., $\pi^p(\omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_p) := \pi^1(\omega_1) \cdots \pi^1(\omega_p)$ for all $\omega_1, \dots, \omega_p \in \Omega^1(\mathcal{A})$, or in other words,

$$\pi^p(a_0\,\delta a_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \delta a_p) = \pi(a_0)[\mathcal{D},\pi(a_1)]\cdots[\mathcal{D},\pi(a_p)]$$

for all $a_0, a_1, \dots, a_p \in \mathcal{A}$.

6.9. Definition ("junk ideal")

Define a graded two-sided ideal of $\Omega^\bullet(\mathcal{A})$ by

$$\mathcal{J}_0 := \bigoplus_{p\geq 0} \mathcal{J}_0^p, \quad \text{where } \mathcal{J}_0^p := \{\omega \in \Omega^p(\mathcal{A}) \mid \pi^p(\omega) = 0\}.$$

We call $\mathcal{J} := \bigoplus_{p\geq 0} \mathcal{J}^p$ with $\mathcal{J}^p := \mathcal{J}_0^p + \delta\mathcal{J}_0^{p-1}$ the junk ideal of $\Omega^\bullet(\mathcal{A})$.

6.10. Proposition:

$\mathcal{J}$ is a two-sided ideal of $\Omega^\bullet(\mathcal{A})$ with $\delta\mathcal{J} \subseteq \mathcal{J}$.

Proof:

Take $\omega = \omega_0 + \delta\omega_1 \in \mathcal{J}$ with $\omega_0 \in \mathcal{J}_0^p$, $\omega_1 \in \mathcal{J}_0^{p-1}$. Then

$$\delta\omega = \delta\omega_0 + \underbrace{\delta^2\omega_1}_{=0} = \delta\omega_0 \in \mathcal{J},$$

which proves $\delta\mathcal{J} \subseteq \mathcal{J}$. Moreover, for $\eta \in \Omega^q(\mathcal{A})$,

$$\begin{aligned}
\omega \otimes_{\mathcal{A}} \eta &= \omega_0 \otimes_{\mathcal{A}} \eta + \delta\omega_1 \otimes_{\mathcal{A}} \eta \\
&= \omega_0 \otimes_{\mathcal{A}} \eta + \delta(\omega_1 \otimes_{\mathcal{A}} \eta) - (-1)^{p-1}\omega_1 \otimes_{\mathcal{A}} \delta\eta \\
&= \underbrace{(\omega_0 \otimes_{\mathcal{A}} \eta + (-1)^p \omega_1 \otimes_{\mathcal{A}} \delta\eta)}_{\in \mathcal{J}_0^{p+q}} + \delta\underbrace{(\omega_1 \otimes_{\mathcal{A}} \eta)}_{\in \mathcal{J}_0^{p+q-1}} \in \mathcal{J}^{p+q},
\end{aligned}$$

and analogously, $\eta \otimes_{\mathcal{A}} \omega \in \mathcal{J}^{p+q}$.

$\square$

6.11. Definition (Connes' forms)

The graded differential algebra $(\Omega_{\mathcal{D}}^{\bullet}(\mathcal{A}), d)$ is defined by

- $\Omega_{\mathcal{D}}^{\bullet}(\mathcal{A}) := \Omega^{\bullet}(\mathcal{A})/\mathcal{J} = \bigoplus_{p\geq 0} \Omega^{p}(\mathcal{A})/\mathcal{J}^{p}$,
- $d = (d^{p})_{p\geq 0}$ with $d^{p}: \Omega_{\mathcal{D}}^{p}(\mathcal{A}) \to \Omega_{\mathcal{D}}^{p+1}(\mathcal{A})$, $[\omega] \mapsto [\delta^{p}\omega]$.

6.12. Remark:

For each $p \geq 0$, we have that

$$\Omega_{\mathcal{D}}^{p}(\mathcal{A}) \cong \pi^{p}(\Omega^{p}(\mathcal{A}))/\pi^{p}(\delta \mathcal{J}_{0}^{p-1}),$$

and thus

$$\Omega_{\mathcal{D}}^{\bullet}(\mathcal{A}) \cong \pi_{\mathcal{D}}(\Omega^{\bullet}(\mathcal{A}))/\pi_{\mathcal{D}}(\delta \mathcal{J}_{0}).$$

6.13. Remark:

In 1986, Connes proved an analytic version of the Hochschild-Kostant-Rosenberg theorem; in fact, he proved that for a compact manifold $\mathcal{M}$, the continuous Hochschild cohomology module $HH^{k}(C^{\infty}(\mathcal{M}), C^{\infty}(\mathcal{M}))$ and the space $\mathcal{D}_{k}(\mathcal{M})$ of de Rham k-currents on $\mathcal{M}$ are isomorphic. A de Rham k-current C is a continuous linear form $\int_{C}: \Omega_{\mathbb{C}}^{k}(\mathcal{M}) \to \mathbb{C}$. The isomorphism

$$HH^{k}(C^{\infty}(\mathcal{M}), C^{\infty}(\mathcal{M})) \longrightarrow \mathcal{D}_{k}(\mathcal{M}), \quad \varphi \mapsto C_{\varphi}$$

is defined by

$$\int_{C_\varphi} a_0\, da_1 \wedge \ldots \wedge da_k := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \varphi(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(k)}),$$

when S_k is the group of all permutations of $\{1, \ldots, k\}$ and $\operatorname{sgn}: S_k \to \{-1, 1\}$ the sign-map. The dual version says that

$$HH_k(C^\infty(M), C^\infty(M)) \cong \Omega^k_{\mathbb{C}}(M).$$

6.14. Theorem:

Let M be a spin^c manifold. Then there exists an isomorphism

$$\sigma_k : \Omega^k_D(A) \longrightarrow \Omega^k_{\mathbb{C}}(M)$$

which commutes with the differentials.

7. Connes' reconstruction theorem

Since the beginnings of noncommutative geometry, it was an open problem to provide a list of sufficient conditions which guarantee that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with a commutative algebra $\mathcal{A}$ comes from a compact oriented smooth manifold M in the sense that $\mathcal{A} \cong C^\infty(M, \mathbb{C})$. This was solved by Connes in 2013.

In the following, let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with $\mathcal{A}$ being commutative.

7.1. Conditions:

(i) Dimension: $(\mathcal{A}, \mathcal{H}, D)$ has dimension $n \in \mathbb{N}$ (in the sense of Remark 4.17 (iii))

(ii) Order one: We have for all $f, g \in \mathcal{A}$

$$[[D, f], g] = 0.$$

(iii) Regularity: $(\mathcal{A}, \mathcal{H}, D)$ is regular (in the sense of Definition 5.4 (i))

(iv) Orientability: There exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$ such that with $\pi_D : \mathcal{A}^{\otimes(n+1)} \to B(\mathcal{H})$, $a_0 \otimes \cdots \otimes a_n \mapsto a_0 [D, a_1] \cdots [D, a_n]$

- $\pi_D(\omega) = \Gamma$, if $(\mathcal{A}, \mathcal{H}, D)$ is even (in the sense of Exercise 3 (iii) on Sheet 4 A B) with the grading $\Gamma \in B(\mathcal{H})$.
- $\pi_D(\omega) = 1$, otherwise

(v) <u>Finiteness and absolute continuity</u>

The space $\mathcal{H}^\infty$ from Definition 5.6 is a finite and projective left $\mathcal{A}$-module, i.e., it can be written as $\mathcal{H}^\infty \cong \mathcal{A}^n e$ with some $e^2 = e = e^* \in M_n(\mathcal{A})$.

$\parallel$ $\{(\xi_1, \dots, \xi_n) \mid \xi_j e_{jk} = \xi_k,\ \xi_j \in \mathcal{A}\}$

Moreover,

$$\langle a\xi, \eta \rangle = \int a\,(\xi|\eta)\,\Delta^{-n/2}$$

for $a \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}_\infty$ defines a hermitian structure $(\cdot|\cdot)$ on $\mathcal{H}_\infty$, i.e., a sesquilinear map

$$(\cdot|\cdot) : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{A}$$

with the property that

- $(a\xi | b\eta) = a(\xi|\eta)b^*$ $\quad \forall a, b \in \mathcal{A}\ \forall \xi, \eta \in \mathcal{H}_\infty$
- $(\xi|\xi) \geq 0$ $\quad \forall \xi \in \mathcal{H}_\infty$
- $(\xi|\xi) = 0 \Rightarrow \xi = 0$

(Indeed, the operators $a\Delta^{-n/2}$, $a \in \mathcal{A}$, are measurable)

7.2. Theorem (Connes reconstruction theorem, 2013)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with $\mathcal{A}$ being commutative and suppose that Conditions 7.1 are satisfied and in addition

- all endomorphisms $T: \mathcal{H}_\infty \to \mathcal{H}_\infty$ of the $\mathcal{A}$-module $\mathcal{H}_\infty$ (i.e., $T(a\xi) = aT(\xi)$ for all $a \in \mathcal{A}, \xi \in \mathcal{H}_\infty$) are <u>regular</u>, i.e., belong to $\mathrm{dom}^\infty(\delta)$.
- The Hochschild cycle c is <u>antisymmetric</u>, i.e.,

$$c = \sum_{i=1}^{N} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_0^i \otimes a_{\sigma(1)}^i \otimes \cdots \otimes a_{\sigma(n)}^i$$

Then there exists a compact oriented smooth manifold M such that $\mathcal{A} \cong C^\infty(M, \mathbb{C})$.