Leibniz rule \( \delta(a_1a_2) = \delta(a_1)a_2 + a_1\delta(a_2) \) for all \( a_1, a_2 \in \mathcal{A} \), then there exists a unique morphism \( \mathbf{g} : \Omega^*(\mathcal{A}) \to \mathcal{B} \) of bimodules such that the diagram

\[
\begin{array}{c}
\Omega^*(\mathcal{A}) \\
\downarrow \mathbf{g} \\
\mathcal{A} \xrightarrow{\delta} \mathcal{B}
\end{array}
\]

commutes.

**Proof:** Analogous to Theorem 6.6. \( \square \)

From now on, consider a neutral triple \((\mathcal{A}, \mathcal{B}, \mathcal{D})\). We apply Theorem 6.8 to the \( \mathcal{A} \)-bimodule \( \mathcal{B}(\mathcal{H}) \) with

\[
a_1 \cdot T \cdot a_2 := \pi(a_1) T \pi(a_2) \quad \forall \, a_1, a_2 \in \mathcal{A}, T \in \mathcal{B}(\mathcal{H})
\]

and the derivation

\[
\mathbf{g} : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \quad a \mapsto [\mathcal{D}, \pi(a)]
\]

This gives a unique morphism \( \pi^1 : \Omega^*(\mathcal{A}) \to \mathcal{B}(\mathcal{H}) \) of bimodules satisfying \( \mathbf{g} = \pi^1 \circ \delta \), i.e., for \( a_0, a_1 \in \mathcal{A} \),

\[
\pi^1(a_0 \delta a_1) = \pi(a_0) \pi^1(\delta a_1) = \pi(a_0) \mathbf{g}(a_1) = \pi(a_0) \pi(a_1)[\mathcal{D}, \pi(a_1)].
\]

We extend \( \pi^1 \) multiplicatively to a morphism

\[
\pi^2 = (\pi^p)_{p \geq 0} : \Omega^*(\mathcal{A}) \to \mathcal{B}(\mathcal{H}),
\]

i.e., \( \pi^p(a_1 \otimes \cdots \otimes a_p) := \pi^1(a_1) \cdots \pi^1(a_p) \) for all \( a_1, \ldots, a_p \in \Omega^*(\mathcal{A}) \), or in other words,
\[ \pi^p(a_0 \, \delta a_1 \, \Theta a_1 \cdots \Theta a_p) = \pi(a_0) [\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_p)] \]

for all \( a_0, a_1, \ldots, a_p \in A \).

6.9. Definition ("junk ideal")

Define a graded two-sided ideal of \( \Omega^*(A) \) by

\[ j_0 := \bigoplus_{p \geq 0} j^p_0, \quad \text{where} \quad j^p_0 := \{ \omega \in \Omega^p(A) \mid \pi^p(\omega) = 0 \}. \]

We call \( j := \bigoplus_{p \geq 0} j^p \) with \( j^p := j^p_0 + \delta j^p_0, \) the junk ideal of \( \Omega^*(A) \).

6.10. Proposition

\( j \) is a two-sided ideal of \( \Omega^*(A) \) with \( \delta j \subseteq j \).

Proof:

Take \( \omega = \omega_0 + \delta \omega_1 \in j \) with \( \omega_0 \in j^p, \omega_1 \in j^{p+q} \). Then

\[ \delta \omega = \delta \omega_0 + \delta \frac{\delta \omega_1}{\omega_0} = \delta \omega_0 \in j, \]

which proves \( \delta j \subseteq j \). Moreover, for \( \gamma \in \Omega^q(A) \),

\[ \omega \otimes_A \gamma = \omega_0 \otimes_A \gamma + \delta \omega_1 \otimes_A \gamma \]

\[ = \omega_0 \otimes_A \gamma + \delta (\omega_1 \otimes_A \gamma) - (-1)^{p-1} \omega_1 \otimes_A \delta \gamma \]

\[ = (\omega_0 \otimes_A \gamma + (-1)^p \omega_1 \otimes_A \delta \gamma) + \delta (\omega_1 \otimes_A \gamma) \in j^{p+q}, \]

and analogously, \( \gamma \otimes_A \omega \in j^{p+q} \).

\[ \square \]
6.11. Definition (Connes' forms)

The graded differential algebra \((\Omega^\cdot\cdot (M), d)\) is defined by

- \(\Omega^\cdot\cdot (M) := \Omega^\cdot (M) / / \oplus_{p \geq 0} \Omega^p (M) / / d_p\),

- \(d = (d_p)_{p \geq 0}\) with \(d_p : \Omega^p (M) \rightarrow \Omega^{p+1} (M), [\omega] \mapsto [d_p \omega]\).

6.12. Remark:

For each \(p \geq 0\), we have that

\[\Omega^p (M) \cong \pi^p (\Omega^0 (M)) / / \pi^p (d_j^{p-1})\]

and thus

\[\Omega^\cdot\cdot (M) \cong \pi^\cdot\cdot (\Omega^0 (M)) / / \pi^\cdot\cdot (d_j^0)\].

6.13. Remark:

In 1986, Connes proved an analytic version of the Hohmwindl–Kontant–Rosenberg theorem; in fact, he proved that for a compact manifold \(M\), the continuous Hochschild cohomology module \(\text{HH}^k (\mathcal{C}^\infty (M), \mathcal{C}^\infty (M))\) and the space \(\mathcal{D}_k (M)\) of de Rham \(k\)-currents on \(M\) are isomorphic. A de Rham \(k\)-current \(\mathcal{C}\) is a continuous linear form \(\mathcal{S}_\mathcal{C} : \Omega^k_c (M) \rightarrow \mathcal{C}\). The isomorphism

\(\text{HH}^k (\mathcal{C}^\infty (M), \mathcal{C}^\infty (M)) \rightarrow \mathcal{D}_k (M), \varphi \mapsto \mathcal{C}_\varphi\)

is defined by
\[
\int_{C^*_t} a_0 \wedge \ldots \wedge a_k := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \Psi(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(k)}),
\]
when \(S_k\) is the group of all permutations of \(\{1, \ldots, k\}\) and \(\text{sgn} : S_k \to \{-1, 1\}\) the signature map. The dual version says that
\[
\mathcal{H}^2_{\mathbb{P}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) \cong \Omega^k_{\mathbb{C}}(M).
\]

6.14. Theorem:

Let \(M\) be a smooth manifold. Then there exists an isomorphism
\[
\sigma_k^h : \Omega^k_{\mathbb{D}}(M) \rightarrow \Omega^k_{\mathbb{C}}(M)
\]
which commutes with the differentials.
Since the beginnings of noncommutative geometry, it was an open problem to provide a list of sufficient conditions which guarantee that a spectral triple \((A, H, D)\) with a commutative algebra \(A\) comes from a compact oriented smooth manifold \(M\) in the sense that \(A \cong C^\infty(M, \mathbb{C})\). This was solved by Connes in 2013.

In the following, let \((A, H, D)\) be a spectral triple with \(A\) being commutative.

7.1. Conditions:

(i) **Dimension**: \((A, H, D)\) has dimension \(n \in \mathbb{N}\)

(\text{in the sense of Remark 4.17 (iii)})

(ii) **Order one**: We have for all \(f, g \in A\)

\[ [[D, f], g] = 0. \]

(iii) **Regularity**: \((A, H, D)\) is regular

(\text{in the sense of Definition 5.4 (i)})

(iv) **Orientability**: There exists a Hochschild cycle \(c \in Z_n(\mathcal{A}, \mathcal{A})\) such that with \(\pi_D: \mathcal{A}^\otimes(n+1) \to \mathcal{B}(H)\)

\[ a_0 \otimes \cdots \otimes a_n \mapsto [D, a_n] \cdots [D, a_0]. \]
\( \pi_D (\omega) = 1 \), if \((\mathcal{A}, \mathcal{M}, \mathcal{D})\) is even (in the
sense of Exercise 3(iii) on Sheet 4 AB) with
the grading \( \mathcal{D} \in \mathcal{B}(\mathcal{H}) \).

\( \pi_D (\omega) = 1 \), otherwise.

\( \text{5. Finite men and absolute continuity} \)

The norm \( \mathcal{H}^{\infty} \) from Definition 5.6 is a finite
and projection left \( \mathcal{A} \)-module, i.e., it can be
written as \( \mathcal{H}^{\infty} = \mathcal{A}^\infty \mathcal{E} \) with some \( \mathcal{E}^2 = \mathcal{E} = \mathcal{E}^* \mathcal{E} \in \mathcal{M}_n (\mathcal{A}) \).
Moreover,
\[
\langle a \xi, \nu \rangle = \int a (\xi | \nu) \Delta^{-\nu/2}
\]
for \( a \in \mathcal{A} \) and \( \xi, \nu \in \mathcal{H}^{\infty} \) defines a Hermitian
structure \( (\cdot | \cdot) \) on \( \mathcal{H}^{\infty} \), i.e., a sesquilinear map

\[
(\cdot | \cdot) : \mathcal{H}^{\infty} \times \mathcal{H}^{\infty} \to \mathcal{A}
\]

with the property that

- \( (a \xi | b \nu) = a (\xi | \nu) \mathcal{E}^* \quad \forall a, b \in \mathcal{A} \forall \xi, \nu \in \mathcal{H}^{\infty} \)
- \( (\xi | \xi) \geq 0 \quad \forall \xi \in \mathcal{H}^{\infty} \)
- \( (\xi | \xi) = 0 \quad \Rightarrow \quad \xi = 0 \)

(Indeed, the operators \( a \Delta^{-\nu/2}, a \in \mathcal{A} \), are measurable.)
7.2 Theorem (Quantum reconstruction theorem, 2013)

Let $(d, \mathcal{N}, d)$ be a spectral triple with $d$ being commutative and suppose that Conditions 7.1 are satisfied and in addition

- all endomorphisms $T : H_\infty \to H_\infty$ of the $d$-module $H_\infty$ (i.e., $T(\alpha \beta) = \alpha T(\beta)$ for all $\alpha \in \mathcal{A}, \beta \in H_\infty$)
  are regular, i.e., belong to $\text{dom}^\infty(\delta)$.

- The Hahn-Banach cycle $c$ is antisymmetric, i.e.,
  \[ c = \sum_{i=1}^{N} \sum_{\sigma \in S_n} \gamma_{\sigma}(\sigma) a_0^i \otimes a_{\sigma(1)}^i \otimes \cdots \otimes a_{\sigma(n)}^i \]

Then there exists a compact oriented smooth manifold $M$ such that $\mathcal{A} \cong C^\infty(M, \mathbb{C})$. 