The proof relies on the following results

5.2. Theorem (Hölder inequality for Dixmier traces):

Let $T, S \in \mathcal{K}(\mathcal{H})$ be given. If we have that

$$T^p, S^q \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$$

for $p, q \in (1,\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\mathrm{Tr}_\omega(|TS|) \leq \mathrm{Tr}_\omega(|T|^p)^{1/p}\,\mathrm{Tr}_\omega(|S|^q)^{1/q}.$$

If we have $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, then

$$\mathrm{Tr}_\omega(|TS|) \leq \|S\|\,\mathrm{Tr}_\omega(|T|).$$

5.3. Theorem:

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple for which $|\mathcal{D}|^{-1}$ is bounded. Then, for any $0 < r < 1$ and each $a \in \mathcal{A}$, $[|\mathcal{D}|^r, \pi(a)]$ is bounded; moreover,

$$\|[|\mathcal{D}|^r, \pi(a)]\| \leq C_r \|[\mathcal{D}, \pi(a)]\| \quad \forall a \in \mathcal{A}$$

for some constant $C_r > 0$.

Proof of Theorem 5.1:

By Theorem 5.2, we get that

$$|\mathrm{Tr}_\omega(\pi(a) T \Delta^{-n/2}) - \mathrm{Tr}_\omega(T\pi(a)\Delta^{-n/2})|$$

$$\overset{(4.3)}{=} |\mathrm{Tr}_\omega(T\Delta^{-n/2}\pi(a)) - \mathrm{Tr}_\omega(T\pi(a)\Delta^{-n/2})|$$

$$= |\mathrm{Tr}_\omega(T[\Delta^{-n/2}, \pi(a)])| \leq \|T\|\,\mathrm{Tr}_\omega(|[\Delta^{-n/2}, \pi(a)]|).$$

It thus suffices to prove that

$$(5.1)\quad \mathrm{Tr}_\omega\left(\left|[|D|^{-n}, \pi(a)]\right|\right) = 0 \qquad \forall a \in \mathcal{A}$$

We choose $r \in (0,1)$ such that $k := \frac{n}{r} \in \mathbb{N}$. Then

$$\begin{aligned}
[|D|^{-n}, \pi(a)] &= [R^k, \pi(a)] \qquad (R := |D|^{-r} \in B(\mathcal{H})) \\
&= \sum_{\ell=1}^{k} R^{\ell-1}[R, \pi(a)] R^{k-\ell} \\
&= -\sum_{\ell=1}^{k} R^{\ell}[R^{-1}, \pi(a)] R^{k-\ell+1} \\
&= -\sum_{\ell=1}^{k} |D|^{-r\ell}[|D|^{r}, \pi(a)] |D|^{-r(k-\ell+1)},
\end{aligned}$$

which yields by Theorem 5.2 and Theorem 5.3 that

$$\begin{aligned}
&\mathrm{Tr}_\omega\left(\left|[|D|^{-n}, \pi(a)]\right|\right) \\
(5.2)\quad &\leq \sum_{\ell=1}^{k} \mathrm{Tr}_\omega\left(\left| |D|^{-r\ell}[|D|^{r}, \pi(a)] |D|^{-r(k-\ell+1)}\right|\right) \\
&\leq \left\|[|D|^{r}, \pi(a)]\right\| \sum_{\ell=1}^{k} \mathrm{Tr}_\omega\left(|D|^{-r\ell p_\ell}\right)^{\frac{1}{p_\ell}} \mathrm{Tr}_\omega\left(|D|^{-r(k-\ell+1)q_\ell}\right)^{\frac{1}{q_\ell}},
\end{aligned}$$

where, for each $\ell = 1, \dots, k$,

$$p_\ell := \frac{2n}{r(2\ell-1)} \quad \text{and} \quad q_\ell := \frac{2n}{r(2k-2\ell+1)}.$$

Since $-r\ell p_\ell < -n$ and $-r(k-\ell+1)q_\ell < -n$, we find that $|D|^{-r\ell p_\ell}$ and $|D|^{-r(k-\ell+1)q_\ell}$ are infinitesimals of order > 1. By Theorem 4.8, the right hand side of (5.2) vanishes. $\square$

5.4. Definition

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.

(i) We say that $(\mathcal{A}, \mathcal{H}, D)$ is <u>regular</u> if the unital complex $*$-algebra $\tilde{\mathcal{A}}_D \subseteq B(\mathcal{H})$ which is generated by $\pi(\mathcal{A})$ and $\{[D, \pi(a)] \mid a \in \mathcal{A}\}$ satisfies

$$\tilde{\mathcal{A}}_D \subseteq \mathrm{dom}^\infty(\delta) := \bigcap_{k=1}^{\infty} \mathrm{dom}(\delta^k)$$

for the unbounded derivation δ given by

$$\delta: B(\mathcal{H}) \supseteq \mathrm{dom}(\delta) \to B(\mathcal{H}), \quad T \mapsto [|D|, T]$$

with $\mathrm{dom}(\delta) := \{T \in B(\mathcal{H}) \mid T \,\mathrm{dom}|D| \subseteq \mathrm{dom}|D|, [|D|, T] \in B(\mathcal{H})\}$; we call $\mathrm{dom}^\infty(\delta)$ the <u>smooth domain</u> of δ. Note that

$$\mathrm{dom}(\delta^k) = \{T \in \mathrm{dom}(\delta^{k-1}) \mid \delta^{k-1}(T) \in \mathrm{dom}(\delta)\}.$$

(ii) Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is (n,∞)-summable for some $n \geq 1$. We say that $(\mathcal{A}, \mathcal{H}, D)$ is <u>tame</u> if

$$\tilde{\tau}: \tilde{\mathcal{A}}_D \to \mathbb{C}, \quad a \mapsto \mathrm{Tr}_\omega(a \Delta^{-n/2}) \tag{5.3}$$

is a trace on $\tilde{\mathcal{A}}_D$ for every Dixmier trace Tr_ω.

In the development of the theory of general spectral triples, the question came up whether every (n,∞)-summable spectral triple is necessarily tame. While this is not true in general, the following theorem provides a criterion, which guarantees tameness in particular for regular spectral triples.

5.5. Theorem (Cipriani, Guido, Scarlatti, 1996):

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an (n,∞)-summable spectral triple and suppose that $\tilde{\mathcal{A}}_{\mathcal{D}} \subseteq \mathrm{dom}(\delta)$. Then

$$\mathrm{Tr}_\omega(a T \Delta^{-n/2}) = \mathrm{Tr}_\omega(T a \Delta^{-n/2})$$

for all $T \in \mathcal{B}(\mathcal{H})$ and $a \in \tilde{\mathcal{A}}_{\mathcal{D}}$, i.e., $\tilde{\tau} : \tilde{\mathcal{A}}_{\mathcal{D}} \to \mathbb{C}$ as defined in (5.3) is a hypertrace on $\tilde{\mathcal{A}}_{\mathcal{D}}$; in particular, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is tame.

Proof:

This follows from Theorem 5.1 as soon as we have shown that $(\tilde{\mathcal{A}}_{\mathcal{D}}, \mathcal{H}, |\mathcal{D}|)$ is an (n,∞)-summable spectral triple. The only questionable part here is that

$$\delta(a) = [|\mathcal{D}|, a] \in \mathcal{B}(\mathcal{H}) \qquad \forall a \in \tilde{\mathcal{A}}_{\mathcal{D}},$$

but this is guaranteed by $\tilde{\mathcal{A}}_{\mathcal{D}} \subseteq \mathrm{dom}(\delta)$.
(Note that $\mathrm{dom}\,|\mathcal{D}| = \mathrm{dom}\,\mathcal{D}$ and that $\mathcal{D}$ has compact resolvents if and only if $(1+\mathcal{D}^2)^{-1}$ is compact.) $\square$

Starting with a regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we can build an abstract pseudodifferential calculus; this is due to Connes and Moscovici (1995).

We present its main ingredients.

5.6. Definition:

For a regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we define the family $(\mathcal{H}^s)_{s\in\mathbb{R}}$ of Sobolev spaces by

$$\mathcal{H}^s := \operatorname{dom} |D|^s \qquad \text{for each } s \in \mathbb{R}.$$

Each space $\mathcal{H}^s$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_s$ given for $\xi, \eta \in \mathcal{H}^\infty$ by

$$\langle \xi, \eta \rangle_s := \langle \xi, \eta \rangle + \langle |D|^s \xi, |D|^s \eta \rangle .$$

Further, we put $\mathcal{H}^\infty := \bigcap_{s\in\mathbb{R}} \mathcal{H}^s$, which is a Fréchet space (i.e., a locally convex Hausdorff space whose topology can be induced by a countable family of seminorms with respect to which it is complete).

5.7. Remark:

For $s, t \in \mathbb{R}$ with $s > t$, we have a continuous inclusion $\mathcal{H}^s \hookrightarrow \mathcal{H}^t$; thus $\mathcal{H}^\infty = \bigcap_{k=0}^\infty \mathcal{H}^k$, which shows that $(\|\cdot\|_k)_{k=0}^\infty$ induces the topology on $\mathcal{H}^\infty$.

5.8. Definition:

For each $r \in \mathbb{R}$, we denote by Op_D^r the vector space of linear operators $T: \mathcal{H}^\infty \to \mathcal{H}^\infty$ for which there are constants C_s for every $s \in \mathbb{R}$ (in fact, $s \in \mathbb{Z}$ suffices!) such that

$$\|T\xi\|_{s-r} \leq C_s \|\xi\|_s \qquad \text{for all } \xi \in \mathcal{H}^\infty,$$

i.e., T extends to a bounded linear operator $T: \mathcal{H}^s \to \mathcal{H}^{s-r}$.

<u>5.9. Theorem</u> (Connes, Moscovici, 1995)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a regular spectral triple, then

$$\tilde{\mathcal{A}}_{\mathcal{D}} \subseteq \mathrm{Op}^0_{\mathcal{D}}$$

and furthermore

$$a - |\mathcal{D}| a |\mathcal{D}|^{-1} \in \mathrm{Op}^{-1}_{\mathcal{D}} \qquad \forall a \in \tilde{\mathcal{A}}_{\mathcal{D}}.$$

<u>Proof</u> (sketch):

① By induction, one checks that for $k \in \mathbb{N}_0$

$$|\mathcal{D}|^k a |\mathcal{D}|^{-k} = \sum_{j=0}^{k} \binom{k}{j} \delta^j(a) |\mathcal{D}|^{-j} \quad \text{and}$$

$$|\mathcal{D}|^{-k} a |\mathcal{D}|^{k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} |\mathcal{D}|^{-j} \delta^j(a),$$

which shows that $|\mathcal{D}|^k a |\mathcal{D}|^{-k}$ is bounded for all $k \in \mathbb{Z}$.

Thus, for each $\xi \in \mathcal{H}^\infty$,

$$\|a\xi\|_k^2 = \|a\xi\|^2 + \| \underbrace{|\mathcal{D}|^k a \xi}_{=(|\mathcal{D}|^k a |\mathcal{D}|^{-k})(|\mathcal{D}|^k \xi)} \|^2$$

$$\leq C_k^2 \|\xi\|_k^2, \quad C_k := \max\{\|a\|, \| |\mathcal{D}|^k a |\mathcal{D}|^{-k}\|\},$$

so that $a \in \mathrm{Op}^0_{\mathcal{D}}$, in fact, for each $a \in \mathrm{dom}^\infty(\delta)$.

② By regularity, we have for each $a \in \tilde{\mathcal{A}}_{\mathcal{D}}$ that $\delta(a) \in \mathrm{dom}^\infty(\delta)$ and thus $\delta(a) \in \mathrm{Op}^0_{\mathcal{D}}$ by ①.

By Definition 5.7, we have further that

$$T |\mathcal{D}|^k \in \mathrm{Op}^{r+k}_{\mathcal{D}} \qquad \forall T \in \mathrm{Op}^r_{\mathcal{D}}.$$

Thus, in summary

$$a - |D| a |D|^{-1} = -\delta(a) |D|^{-1} \in Op_D^{-1}$$

for each $a \in \tilde{\mathcal{A}}_D$. $\square$

5.10. Definition:

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple.

(i) Elements in the algebra $\Psi^0(\mathcal{A})$ which is generated by

$$\bigcup_{k=0}^{\infty} \{ \delta^k(\pi(a)), \delta^k([D, \pi(a)]) \mid a \in \mathcal{A} \}$$

are called pseudodifferential operators of order 0.

(ii) A pseudodifferential operator of order d, $d \in \mathbb{Z}$, is an operator $P: \mathcal{H}^\infty \to \mathcal{H}^\infty$ of the form

$$P \simeq a_d |D|^d + a_{d-1} |D|^{d-1} + \dots$$

with $a_d, a_{d-1}, \dots \in \Psi^0(\mathcal{A})$, i.e.,

$\forall N \; \exists R: \quad P = \sum_{-N \le k \le d} a_k |D|^k + R$ and $R |D|^N \in \mathrm{dom}^\infty(\delta)$;

we denote the space of all those operators by $\Psi^d(\mathcal{A})$.

5.11. Remark:

By Theorem 5.9 and its proof, we have that

$$\tilde{\mathcal{A}}_D \subseteq \Psi^0(\mathcal{A}) \subseteq Op_D^0$$

and furthermore $\Psi^k(\mathcal{A}) \subseteq Op_D^k$ for each $k \in \mathbb{Z}$.