Then, $\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H})_+ \to [0,\infty)$ satisfies the conditions

- $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2) \quad \forall T_1, T_2 \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$,
- $\mathrm{Tr}_\omega(\lambda T) = \lambda \mathrm{Tr}_\omega(T) \quad \forall \lambda \geq 0, T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$,

and extends uniquely to a positive linear map

$$\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C},$$

which is a hypertrace, i.e.,

(4.3) $\quad \mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS) \quad \forall S \in \mathcal{B}(\mathcal{H}), T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$,

and satisfies

(4.4) $\quad \mathrm{Tr}_\omega(T) = 0 \quad \forall T \in \mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^{(1,\infty)}(\mathcal{H})$
(note that $\bigcup_{\alpha>1} I_\alpha(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H})$)

We call $\mathrm{Tr}_\omega$ a Dixmier trace.

Proof:

For every $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$, $(\gamma_N(T))_{N=1}^\infty$ is bounded (see Remark 4.6); hence, $\mathrm{Tr}_\omega(T)$ is well-defined and clearly $\mathrm{Tr}_\omega(T) \geq 0$ by (i). For $T_1, T_2 \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$, it follows from Proposition 4.7 and property (i) of $\omega$ that

$$\mathrm{Tr}_\omega(T_1 + T_2) \leq \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$$

$$\leq \omega\Big(\big(\gamma_{2N}(T_1+T_2)\big)_{N=1}^\infty\Big) + \omega\Big(\Big(\frac{\log(2)}{\log(N)}\gamma_{2N}(T_1+T_2)\Big)_{N=1}^\infty\Big)$$

By property (ii) of $\omega$, we get that

$$\omega\Big(\Big(\frac{\log(2)}{\log(N)}\gamma_{2N}(T_1+T_2)\Big)_{N=1}^\infty\Big) = \lim_{N\to\infty} \frac{\log(2)}{\log(N)} \underbrace{\gamma_{2N}(T_1+T_2)}_{\text{bounded}} = 0,$$

and property (iii) yields that

$$\omega\left(\left(\gamma_{2N}(T_1+T_2)\right)_{N=1}^{\infty}\right) = \omega\left(\left(\gamma_N(T_1+T_2)\right)_{N=1}^{\infty}\right) = \mathrm{Tr}_\omega(T_1+T_2).$$

In summary, we get that $\mathrm{Tr}_\omega(T_1+T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$.
That $\mathrm{Tr}_\omega(\lambda T) = \lambda \mathrm{Tr}_\omega(T)$ for all $\lambda \geq 0$ and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$ is clear since $\omega$ is linear and $\gamma_N(\lambda T) = \lambda\gamma_N(T)$ for each $N \in \mathbb{N}$.

The extension of $\mathrm{Tr}_\omega$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ uses that

- each $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ can be written uniquely as $T = \mathrm{Re}(T) + i\,\mathrm{Im}(T)$ with selfadjoint $\mathrm{Re}(T), \mathrm{Im}(T) \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$;
- each $T = T^* \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ can be written as $T = P_+|T|P_+ - P_-|T|P_-$ for projections $P_+, P_- \in vN(T)$ satisfying $P_+ + P_- = 1$ (using the measurable functional calculus $\tilde{\Phi}: B_b(Sp(T)) \to vN(T)$ from Remark 6.12 (iv), FA II, with $P_+ := \tilde{\Phi}(\chi_{[0,\infty)})$ and $P_- := \tilde{\Phi}(\chi_{(-\infty,0)})$), where $P_+|T|P_+, P_-|T|P_- \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$ due to Remark 4.3. since $|T| \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$.

To prove (4.3), we note first that (4.1) implies for $n \in \mathbb{N}_0$

$$\mu_n(UTU^*) = \mu_n(T) \quad \forall\, T \in K(\mathcal{H}),\ U \in B(\mathcal{H}) \text{ unitary},$$

which yields by definition for $N \in \mathbb{N}$

$$\gamma_N(UTU^*) = \gamma_N(T) \quad \forall\, T \in K(\mathcal{H}),\ U \in B(\mathcal{H}) \text{ unitary}$$

and hence

$$\mathrm{Tr}_\omega(UTU^*) = \mathrm{Tr}_\omega(T) \quad \forall\, T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}),\ U \in B(\mathcal{H}) \text{ unitary},$$

or equivalently, since $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$,

$$\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU) \quad \forall\, T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}),\ U \in B(\mathcal{H}) \text{ unitary}.$$

Since every $S \in B(\mathcal{H})$ is a linear combination of (in fact four) unitaries (see the proof of Lemma 6.13, FA II), the latter yields (4.3).

To verify (4.4), we take $T \in \mathcal{L}^1(\mathcal{H})$ and note that $(\sigma_N(T))_{N=1}^\infty$ is bounded by $\|T\|_1$ due to (4.2); thus, $\gamma_N(T) \to 0$ as $N \to \infty$, so that

$$\mathrm{Tr}_\omega(T) \underset{\substack{\uparrow \\ \text{w.l.o.g. } T \geq 0}}{=} \omega\big((\gamma_N(T))_{N=1}^\infty\big) \overset{\text{(ii)}}{=} \lim_{N\to\infty} \gamma_N(T) = 0.$$

Note that $I_\alpha(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H})$ for every $\alpha > 1$, since for each $T \in I_\alpha(\mathcal{H})$, we find $C < \infty$ such that

$$\|T\|_1 \overset{(4.2)}{=} \sum_{n=0}^\infty \mu_n(T) \leq \|T\| + C \sum_{n=1}^\infty n^{-\alpha} < \infty.$$

$\square$

4.9. Definition:

Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. We say that $T$ is <u>measurable</u> if the value of $\mathrm{Tr}_\omega(T)$ is independent of $\omega$; we denote this common value by $⨍ T$ and call it the <u>noncommutative integral of $T$</u>. Moreover, we put

$$\mathcal{M}(\mathcal{H}) := \{T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) \mid T \text{ measurable}\}.$$

4.10 Remark:

(i) The existence of (in fact infinitely many) linear maps

$\omega : \ell^\infty(\mathbb{N},\mathbb{R}) \to \mathbb{R}$ satisfying the conditions (i), (ii), and (iii) in Theorem 4.8 was proved by Dixmier (1966): With the construction of $\mathrm{Tr}_\omega$, he proved the existence of <u>singular traces</u> on $B(\mathcal{H})$ (i.e., traces that vanish on $\mathcal{L}^1(\mathcal{H})$) and settled to the negative the question of the uniqueness of the trace on $B(\mathcal{H})$.

(ii) An alternative approach was developed by Connes. It relies on the (piecewise linear) interpolation of $(\sigma_N(T))_{N=1}^\infty$ given by

$$\sigma_\lambda(T) := \inf\{\|R\|_1 + \lambda\|S\| \mid R \in \mathcal{L}^1(\mathcal{H}), S \in K(\mathcal{H}) : T = R + S\}$$

for each $\lambda > 0$. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ and any $a > e$,

$$\gamma : [a,\infty) \to \mathbb{R}, \quad \lambda \mapsto \frac{\sigma_\lambda(T)}{\log(\lambda)}$$

is a continuous and bounded function; its <u>Cesàro mean</u> with respect to the Haar measure $\frac{du}{u}$ on the multiplicative group $(0,\infty)$ is given by

$$\tau_\lambda(T) := \frac{1}{\log(\lambda)} \int_a^\lambda \gamma(u)\,\frac{du}{u} \qquad \text{for each } \lambda \in [a,\infty)$$

and defines a function $\lambda \mapsto \tau_\lambda(T)$ in $C_b([a,\infty))$. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+$, let $\dot{\tau}(T)$ be the class of $\lambda \mapsto \tau_\lambda(T)$ in the quotient $C^*$-algebra $\mathcal{B} := C_b([a,\infty)) / C_0([a,\infty))$.

One can show that $\dot{\tau} : \mathcal{L}^{(1,\infty)}(\mathcal{H})_+ \to \mathcal{B}$ extends to a positive linear map $\dot{\tau} : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to \mathcal{B}$ satisfying

$$\dot{\tau}(ST) = \dot{\tau}(TS) \quad \text{for all } T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}),\ S \in \mathcal{B}(\mathcal{H}).$$

For every state $\omega$ on $\mathcal{B}$, one defines $\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C}$ by

$$\mathrm{Tr}_\omega(T) := \omega(\dot{\tau}(T)) \quad \text{for all } T \in \mathcal{L}^{(1,\infty)}(\mathcal{H});$$

it satisfies (4.3) and (4.4) in Theorem 4.8.
Moreover, we have that $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is measurable if and only if $\lim_{\lambda\to\infty} \tau_\lambda(T)$ exists, in which case

$$⨍ T = \lim_{\lambda\to\infty} \tau_\lambda(T).$$

Note that exhibiting a state $\omega$ on the (non-separable) $C^*$-algebra $\mathcal{B}$ requires the axiom of choice.

(iii) $\mathcal{M}(\mathcal{H})$ is a vector space and satisfies

$$STS^{-1} \in \mathcal{M}(\mathcal{H}) \qquad \forall S \in \mathcal{B}(\mathcal{H}) \text{ invertible},\ T \in \mathcal{M}(\mathcal{H}).$$

Moreover, one can check that $\mathcal{M}(\mathcal{H}) \subsetneqq \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

(iv) With the help of <u>real interpolation theory</u>, one can construct out of $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ a family

$$\mathcal{L}^{(p,q)}(\mathcal{H}) \qquad \text{with } 1 < p < \infty \text{ and } 1 \le q \le \infty$$

of two-sided ideals in $\mathcal{B}(\mathcal{H})$. In fact, $\mathcal{L}^{(p,q)}(\mathcal{H})$ for $q < \infty$ consists of those $T \in \mathcal{K}(\mathcal{H})$ which satisfy

$$\sum_{N=1}^{\infty} N^{(\frac{1}{p}-1)q-1}\,\sigma_N(T)^q < \infty,$$

while $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ consists of those $T \in K(\mathcal{H})$ for which

$$\|T\|_{(p,\infty)} := \sup_{N\in\mathbb{N}} N^{\frac{1}{p}-1}\,\sigma_N(T) \leq \infty;$$

it follows that $\mathcal{L}^{(p,\infty)}(\mathcal{H}) = I_{1/p}(\mathcal{H})$ for each $p>1$.

On the diagonal, one finds the <u>Schatten-ideals</u>

$$\mathcal{L}^p(\mathcal{H}) := \mathcal{L}^{(p,p)}(\mathcal{H}),$$

where the interpolation norm $\|\cdot\|_{(p,p)}$ on $\mathcal{L}^{(p,p)}(\mathcal{H})$ is equivalent to the <u>Schatten</u> p-<u>norm</u>

$$\|T\|_p := \mathrm{Tr}\,(|T|^p)^{1/p} \quad \text{for} \quad T \in \mathcal{L}^p(\mathcal{H}).$$

A noncommutative integration theory for which $\mathcal{L}^p(\mathcal{H})$ serves as an analogue of the $L^p$-space in Lebesgue integration theory, was developed by Segal in the 50's.

**4.11. Definition**

Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple. We say that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is

(i) <u>p-summable</u> if $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^p(\mathcal{H})$;

(ii) <u>$(p,\infty)$-summable</u> if $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$;

(iii) <u>$\Theta$-summable</u> if $e^{-t\mathcal{D}^2} \in \mathcal{L}^1(\mathcal{H})$ for all $t>0$.